ENG

THE MATHEMATICAL GRAMMAR SCHOOL CUP

- MATHEMATICS -

26. June 2023.

This test consists of 12 problems on two pages. The problems are divided into two parts: multiple-choice questions and "fill-in" problems where a student should fill in their answers by hand on the answer sheet. The examination lasts 180 minutes. The use of calculators, computers, or other electronic devices is strictly prohibited.

PART ONE

Problems 1 to 8 are multiple-choice problems. Out of the five choices offered for a problem, exactly one is the correct answer. On the answer sheet, you should circle only the letter that corresponds to the answer you have chosen. Each correct answer is worth 5 points.

1. The function $f:\mathbb{N}\to\mathbb{N}$ satisfies $f(f(m)+f(n))=m+n$, for all $m,n\in\mathbb{N}$. The total number of all integer divisors (not necessarily positive) of the number $f(f(2023))$ is:

(A) 4 (B) 20 (C) 8 (D) 12 (E) 6.

2. Let ABC be an acute angled triangle with $\sphericalangle BAC=60^\circ$ and $AB>AC$. Let I be the incenter and H the orthocenter of the triangle ABC . Then $\sphericalangle AHI$ is equal:

(A) $\frac{3\sphericalangle ABC}{4}$ (B) $\frac{3\sphericalangle ABC}{2}$ (C) $\frac{\sphericalangle ABC}{2}$ (D) $\frac{4\sphericalangle ABC}{3}$ (E) $\frac{\sphericalangle ABC}{3}$.

3. Let x and y be positive integers and let p be a prime number. The number of elements of the set

$$\{(y-x)p+2023\,:\,p^x-1=y^3\}$$

is:

(A) 3 (B) 2 (C) 1 (D) 0 (E) ∞.

4. The number of words with 11 digits that can be formed from the alphabet $\{0,1,2,3,4\}$ (words can start with zero) such that the neighboring digits differ by exactly one is:

(A) 810 (B) 648 (C) 243 (D) 1134 (E) 1053.

5. In a triangle ABC, points M and N are on sides AB and AC, respectively, such that $MB=BC=CN$. Let R and r denote the circumradius and the inradius of the triangle ABC, respectively. If the ratio $\dfrac{r}{R}$ equals 0.125, then the ratio MN/BC is:

(A) $\frac{4}{5}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{2\sqrt{3}}{3}$ (D) $\frac{1}{2}$ (E) $\frac{\sqrt{3}}{2}$.

6. Let $f(x)=ax^2+bx+c$, with $a,\,b,\,c\in\mathbb{R}$. If $|f(0)|\leqslant 1, |f(1)|\leqslant 1, |f(-1)|\leqslant 1$, then $\max\limits_{x\in[-1,1]}|f(x)|$ is:

(A) 0.8 (B) 1 (C) 1.25 (D) 1.5 (E) 2.

7. For a positive integer n denote by $a(n)$ its greatest odd divisor. If

$$b=a(1012)+a(1013)+...+a(2023),$$

then the sum of the digits of the largest positive four-digit divisor of $b-1$ is:

(A) 15 (B) 24 (C) 33 (D) 9 (E) 5.

8. Let $ABCD$ be a rectangle with $AB=30$ and $BC=60$. Let k be the circle whose diameter is AD and l be the circle whose diameter is AB. Let circles k and l meet each other again in P. Let AP intersect

BC at E. If F is the point on AB different from B such that EF is tangent to the circle k, then the area of triangle AEF is:

(A) 125 (B) 100 (C) 90 (D) 75 (E) 60.

PART TWO

Problems 9 to 12 are "fill-in" problems. Points for a certain part of the problem will be awarded only if this and all the answers to the previous parts are correct.

9. Find all positive integers a, b, c such that:

$$2^a + 15^b = c^2.$$

10. Let n be an integer. Function $f : \mathbb{Z} \to \mathbb{Z}$ satisfies $f(-f(x) - f(y)) = n - x - y$ for all $x, y \in \mathbb{Z}$. Find the number of such functions, depending on n.

11. How many crosses (see picture) can be placed without an overlap on a $8 \cdot 8$ board, and how many on a $7 \cdot 7$ board? Crosses can be rotated.

12. Let ABC be a triangle, and let the incircle touch BC, CA, AB at D, E, F, respectively. Let P be a point on the incircle that satisfies $\sphericalangle CPE + \sphericalangle BPF = 180$. Prove P lies on a midline of triangle DEF.

GOOD LUCK!!!

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