The Mathematical Grammar School Cup

≈ MATHEMATICS ≈

This test consists of 12 problems on three pages. The problems are divided into two parts: multiple choice questions and "fill-in" problems where a student should fill in their answers by hand on the answer sheet. The examination lasts 180 minutes. The use of calculators, computers, or other electronic devices is strictly prohibited!

PART ONE

Problems 1 to 8 are multiple choice problems. Out of the six choices offered for a problem, exactly one is the correct answer. On the answer sheet you should circle only the letter that corresponds to the answer you have chosen. Each correct answer is worth 5 points.

1. Let x, y, and z be different real numbers so that $x^2 - yz = y^2 + xz = z^2 + xy = 7$. Then $x^2 + y^2 + z^2$ is equal to:

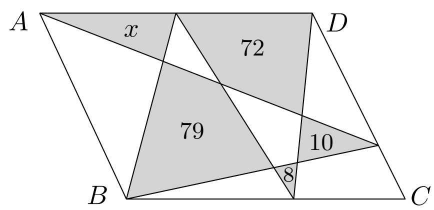
A) 15 B) 14 C) 12 D) 21 E) 7 F) 9

2. How many positive integers from 1 to 2021, inclusive, are divisible by 6 or 9, but not by both?

A) 486 B) 560 C) 392 D) 336 E) 523 F) 448

3. In the picture to the right, $ABCD$ is a paralellogram with line segments that divide it up into various regions. The area of some of the regions are given. Then x is equal to:

A) 12 B) 10 C) 13 D) 8 E) 11 F) 9



4. Once upon a time, a team of students went to another country for a mathematics competition. They stayed in a hotel where each floor had 10 rooms numbered by consecutive integers, starting from 10 – rooms 10 throughout 19 were on the first floor, rooms 20 to 29 on the second floor, rooms 30 to 39 on the third, and so on. Two team members, Igor and Pavle, realized that Pavle is staying on the n^{th} floor, where n is Igor's room number, and that the sum of their room numbers is 260. What is the absolute value of the difference of their room numbers?

A) 198 B) 292 C) 258 D) 214 E) 246 F) 310

5. Aleksa wants to color each unit square of a 2×2021 grid of 1×1 squares in such a way that:

(a) each of the 4042 squares is colored either yellow, red, or blue; **and**

(b) whenever two squares are horizontally or vertically adjacent, they must have different colors.

In how many different ways can Aleksa do this?

A) 2^{2020} B) 3^{2021} C) $2 \cdot 3^{2021}$ D) $3 \cdot 2^{2020}$ E) 3^{2020} F) $3 \cdot 2^{2021}$

6. For how many integers n is $\dfrac{n}{150-n}$ the square of an integer?

A) 6 B) 4 C) 7 or more D) 2 or less E) 5 F) 3

7. Let ABC be a triangle with side lengths $1,2$ cm, $1,6$ cm, and 2 cm, and let F be the set off all points P of the plane ABC that are within distance 1 cm from at least two points from the set $\{A, B, C\}$. The area of set F is equal to (in cm^2):

A) $\dfrac{\pi}{4} + 0,96$ B) $\dfrac{\pi}{4} + 0,92$ C) $\pi - 1,8$ D) $\dfrac{\pi}{2} - 0,9$ E) $\pi - 1,92$ F) $\dfrac{\pi}{2} - 0,96$

8. For a positive integer n, let $\text{odd}(n)$ be the largest odd factor of n. Fore example, $\text{odd}(16) = 1$ and $\text{odd}(30) = 15$. Then the sum $\text{odd}(119) + \text{odd}(120) + \text{odd}(121) + \cdots + \text{odd}(235) + \text{odd}(236)$ is equal to:

A) 13689 B) 14161 C) 13924 D) 13790 E) 13456 F) 14120

PART TWO

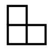
Problems 9 to 12 are "fill-in" problems. Points for a certain part of the problem will be awarded only if this and ***all*** *the answers to the previous parts are correct.*

9. Let $ABCD$ be a trapezoid such that $AB \parallel CD$ and $AB = 23$, $BC = 5$, $CD = 10$. Furthermore, $\sphericalangle DCB = 90^\circ + \sphericalangle BAD$.

(a) *[4 points]* Compute the length of side AD.

(b) *[5 points]* Let M and N be the midpoints of segments AB and CD, respectively. Compute the length of segment MN.

(c) *[6 points]* Compute the length h of the altitude of this trapezoid.

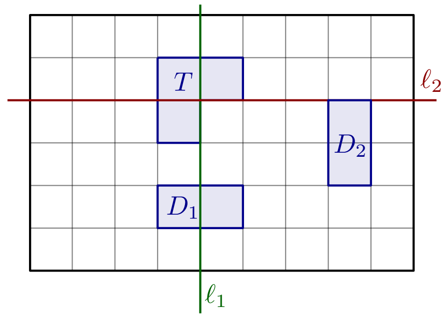
10. A *tiling* of a board is a way to place several tiles on that board so that all of its squares are covered, but no tiles overlap or lie partially off the board. It is allowed to rotate tiles. In this problem we will consider tilings by *dominoes* () or L–shaped *trominoes* ().

(a) */3 points/* How many different tilings of a 2×10 board by dominoes are there?

(b) */4 points/* How many different tilings of a 3×100 board by trominoes are there?

(c) */4 points/* Let $n \times m$, for $2 \leqslant n \leqslant m$, be the board of the smallest area which can be tiled by dominoes in such a way that every line parallel to the board sides, that intersects the interior of the board, also intersects the interior of *at least one domino*. What are n and m equal to?

(d) */4 points/* Draw at least one tiling of a 4×9 board by trominoes in such a way that that every line parallel to the board sides, that intersects the interior of the board, also intersects the interior of *at least one tromino*.



For example, in the picture above line ℓ_1 intersects the interior of both domino D_1 and tromino T, while ℓ_2 intersects only the interior of tromino T.

11. Let $S(n)$ be the sum of digits of a positive integer n.

(a) */2 points/* Find the smallest positive integer n such that $9 \mid S(n)$ and $9 \mid S(n+1)$. If no such number exists, write 'X'.

(b) */5 points/* Find the smallest positive integer n such that $11 \mid S(n)$ and $11 \mid S(n+1)$. If no such number exists, write 'X'.

(c) */8 points/* Finally, find the number of positive integers n with at most 8 digits for which it holds that $11 \mid n$, $11 \mid S(n)$, and $11 \mid S(n+1)$, and write all of them.

12. (a) */1 point/* Find the minimum m_a of the expression $\dfrac{x^2+1}{x}$, for $x > 0$.

(b) */4 points/* Find the minimum m_b of the expression $\dfrac{x^3+3x+9}{x^2}$, for $x > 0$.

(c) */4 points/* Find the minimum m_c of the expression $\dfrac{(x+1)(y+2)(xy+2)}{xy}$, for $x > 0$ and $y > 0$.

(d) */6 points/* Find the minimum m_d of the expression $\dfrac{(x+4)(y+1)(xy+864)}{xy}$, for $x > 0$ and $y > 0$.

End of exam