ENG

THE MATHEMATICAL GRAMMAR SCHOOL CUP
-MATHEMATICS-

Belgrade, June 26, 2019

The problems are divided into two parts: multiple choice questions and problems in the standard form. The students should use separate sections on the notebook provided for each of the two parts. The examination lasts 180 minutes.

PART ONE

Problems 1 to 8 are multiple choice problems. Out of the five choices offered for a problem, exactly one is the correct answer. On the answer sheet you should circle only the letter that corresponds to the answer you have chosen. Each correct answer is worth 5 points.

1. Which of the following is equal to $\left(\dfrac{1}{3\sqrt{2}-4}-\dfrac{1}{3\sqrt{2}+4}\right):\left(\dfrac{1}{5+2\sqrt{6}}+5-2\sqrt{6}\right)$?

(A) $10+4\sqrt{6}$; (B) $\dfrac{8}{5+2\sqrt{6}}$; (C) $\dfrac{2}{5}$; (D) $\dfrac{\sqrt{2}+\sqrt{6}}{3}$; (E) 12.

2. The price of a book is increased by 20%, and afterwards the new price is increased by 35%. The total increase in price is equal to:

(A) 55%; (B) 60%; (C) 62%; (D) 65%; (E) 70%.

3. The product of all solutions of the equation $x+2\cdot|x-4|=7$ in the set of real numbers is equal to:

(A) 6; (B) 5; (C) 1; (D) -1; (E) -6.

4. In triangle ABC, angle ACB is 90° and point H is the foot of the altitude from vertex C to side AB. If $BC=5$ cm and $BH=1$ cm, then $\dfrac{CH}{CA}$ is equal to:

(A) $2\sqrt{6}$; (B) 5; (C) $\dfrac{1}{10\sqrt{6}}$; (D) $\dfrac{1}{2\sqrt{6}}$; (E) $\dfrac{1}{5}$.

5. Let a,b,c be positive real numbers so that $a+c=2b$ and $c^2+ac-a^2=b^2$. Which of the following is equal to $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}$?

(A) $3\sqrt{3}$; (B) 3; (C) $3\sqrt{2}$; (D) 6; (E) $2\sqrt{3}$.

6. Let $0 < t < 1$ be the real number such that $t + \frac{1}{t} = 7$. The value of $\sqrt{t} - \frac{1}{\sqrt{t}}$ is equal to:

(A) $\sqrt{5}$; (B) $-\sqrt{5}$; (C) $\sqrt{7}$; (D) $-\sqrt{7}$; (E) 0.

7. Let $ABCD$ be a parallelogram and $\sphericalangle DAB = 30^\circ$. If $AB = 4\sqrt{3}$ cm and if the area of the parallelogram is equal to $20\sqrt{3}$ cm^2, then the length of AC is equal to:

(A) $2\sqrt{\frac{91}{3}}$ cm; (B) $2\sqrt{7}$ cm; (C) $\sqrt{138}$ cm; (D) $2\sqrt{67}$ cm; (E) $2\sqrt{23}$ cm.

8. Let $A = \{a_1, a_2, \ldots, a_{11}\}$ be a set. How many pairs (X, Y) are there, so that $X \subset A$, $Y \subset A$, and the number of elements in the sets is: $|X| = 8$, $|Y| = 7$, and $|X \cap Y| = 5$?

(A) 9240; (B) 3245; (C) 27720; (D) 87512; (E) 13860.

PART TWO

Problems 9 to 12 are problems in the standard form, and solutions have to be written and explained in detail. Each complete solution to a problem is worth 15 points.

9. Determine the last 3 digits of number $b = 1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot 26062019$.

10. Let k be a circle and let AC and BD be two chords of different lengths which intersect in point G (A, B, C, D are distinct points). Let H be the foot of the perpendicular from point G to line segment AD. Line GH intersects line segment BC at point P so that $BP = PC$. Prove that $AC \perp BD$.

11. Aleksa and Paja wrote 2019 positive integers on a blackboard. In one step, one can erase any two numbers a and b from the blackboard, and write (a, b) and $[a, b]$ instead (here, (a, b) denotes the greatest common divisor of a and b, and $[a, b]$ denotes their least common multiple). Prove that there exists a positive integer n such that, after n steps, the collection of numbers written on the blackboard cannot be changed anymore by using the procedure described above (the order in which the numbers are written on the blackboard is of no importance).

12. Find all triples (a, b, c) of real numbers so that:

$$\{a, b, c\} = \{ab + a + b, bc + b + c, ca + c + a\}.$$