THE MATHEMATICAL GRAMMAR SCHOOL CUP
-MATHEMATICS-
Belgrade, June 27, 2017

PART ONE

Problems 1 to 8 are multiple choice problems. Out of five choices offered for a problem, exactly one is the correct answer. On the answer sheet you should circle only the letter that corresponds to the answer you have chosen. Each correct answer is worth 5 points.

1. If number a is increased by $10\,\%$, and number b is decreased by $10\,\%$, the resulting numbers will be equal to each other. What is the ratio of a to b?

(A) $\frac{9}{11}$; (B) $\frac{9}{10}$; (C) $\frac{1}{1}$; (D) $\frac{11}{9}$; (E) $\frac{10}{9}$.

2. The bases of a right-angled trapezium have lengths equal to 5 cm and 2 cm, and the acute angle is equal to $45°$. The area of this trapezium is equal to:

(A) 9 cm^2; (B) 10 cm^2; (C) $12,5$ cm^2; (D) 7 cm^2; (E) $10,5$ cm^2.

3. The sum of all solutions of the equation $||3x - 2| - x| = 2$ in the set of real numbers is equal to:

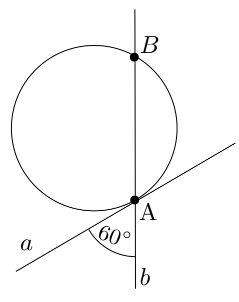
(A) 0; (B) 2; (C) 3; (D) 1; (E) 4.

4. How many integers m are there such that the number $\dfrac{2m+12}{m+3}$ is also an integer?

(A) 5 or less; (B) 6; (C) 7; (D) 8; (E) 9 or more.

5. Two lines a and b passing through point A intersect each other at an angle of $60°$. A circle, which is tangent to line a at point A, intersects line b in point B as well, as shown in the figure to the right. If $AB = 12$ cm, the area of the circle is equal to:

(A) 144π cm^2; (B) 36π cm^2; (C) 48π cm^2;
(D) 54π cm^2; (E) 72π cm^2.



6. The number of odd 3-digit positive integers with distinct digits is equal to:

(A) 270; (B) 300; (C) 315; (D) 320; (E) 360.

7. In triangle ABC, BE and CF are perpendicular to the angle BAC bisector, denoted by ℓ (where E, F are points on ℓ). If $AB = 8$ cm, $AC = 12$ cm, and $AF = 9$ cm, the length of segment AE is equal to:

(A) 6 cm; (B) $4\sqrt{3}$ cm; (C) 5 cm; (D) $\dfrac{32}{3}$ cm; (E) $3\sqrt{3}$ cm.

8. The number of all right-angled triangles, with the property that one cathetus (leg) has length 21, while both the other cathetus (leg) and the hypotenuse have integer lengths, is equal to:

(A) 2; (B) 1 or less; (C) 5 or more; (D) 3; (E) 4.

PART TWO

Problems 9 to 12 are classic problems, and solutions have to be written and explained in details. Each completely correct solution to a problem is worth 15 points.

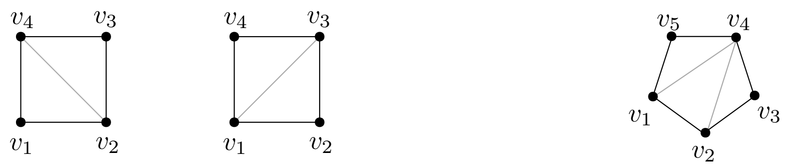
9. Let k be the circumcircle of triangle ABC, and let D, E, and F be the midpoints of those arcs BC, AC, AB of k, that do not contain points A, B, C, respectively. If $P = AB \cap DF$ and $Q = AC \cap DE$, prove that PQ is parallel to BC.

10. Let a, b, c, and m be integers and $m \geq 2$. If $a^n + bn + c$ is divisible by m for all positive integers n, prove that b^2 is divisible by m. Does b always have to be divisible by m?

11. If x, y, z are positive real numbers such that $x + y + z = 1$. Prove the following inequality:

$$xy + yz + zx - xyz \leq \frac{8}{27}.$$

12. In how many ways can a convex 2017-gon be divided into triangles by 2014 diagonals that do not intersect each other (except possibly in their endpoints), in such a way that each triangle has at least one edge in common with that 2017-gon? The answer has to be given in the form $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, for some positive integer k, prime numbers $p_1, p_2, \ldots, p_k$, and positive integers $\alpha_1, \ldots, \alpha_k$.



EXAMPLE. On the left: For a rectangle there are exactly two such divisions.
On the right: One of the divisions of a 5-gon satisfying the conditions.