

THE MATHEMATICAL GRAMMAR SCHOOL CUP
-MATHEMATICS-
BELGRADE, June 23, 2015

PART ONE

Problems 1 to 8 are multiple choice problems. Out of five offered choices for a problem, exactly one is the correct answer. On the answer sheet you should circle only the letter that corresponds to the answer you have chosen.

1. What is the last digit of the number 333^{33} ?
(A) 1; (B) 3; (C) 5; (D) 7; (E) 9.

2. If x and y are positive real numbers such that $x^2 + y^2 = 87$ and $xy = 30$, then their sum $x + y$ is equal to:
(A) $3\sqrt{7}$; (B) $\sqrt{87}$; (C) $7\sqrt{3}$; (D) $8\sqrt{5}$; (E) $10\sqrt{2}$.

3. If numerator a of the quotient $\frac{a}{b}$ ($b \neq 0$) increases 20%, and the denominator b decreases 20%, by how many percent will the quotient increase?
(A) 40%; (B) 20%; (C) 25%; (D) 36%; (E) 50%.

4. The equation $|2x + 5| = x + 5$ has:
(A) exactly one solution and it is negative; (B) exactly one solution and it is positive;
(C) exactly one solution and it is equal to zero; (D) two solutions; (E) no solutions.

5. The areas of three faces of a cuboid (rectangular parallelepiped) are equal to 12, 8, and 6. The volume of that cuboid is:
(A) 24; (B) 26; (C) 42; (D) 44; (E) 48.

6. The height lengths of a triangle are equal to 12, 15, and 20. The area of that triangle is equal to:
(A) 84; (B) 90; (C) 120; (D) 150; (E) 180.

7. The number of all pairs (m, n) of positive integers m and n such that $(m-8)(m-10) = 2^n$ is equal to:
 (A) zero; (B) one; (C) two; (D) three; (E) four or more.
8. In a right-angled trapezoid $ABCD$, with right angle at the vertex A , the diagonals AC and BD are perpendicular to each other. If the bases of the trapezoid are $AB = 9$ and $CD = 4$, the length of the side AD is equal to:
 (A) 5; (B) $5\sqrt{2}$; (C) 6; (D) $5\sqrt{3}$; (E) 8.

PART TWO

9. Find all positive integers n such that the last digit (in the decimal representation) of the sum $1 + 2 + 3 + \cdots + n$ is equal to 7.
10. Let A_1 , B_1 , and C_1 be the touching points of the inscribed circle of a triangle ABC with the sides $BC = a$, $AC = b$, and $AB = c$, respectively, and let $AC_1 = p$, $BA_1 = q$, and $CB_1 = r$. Prove that the following inequality holds:

$$\frac{p}{a} + \frac{q}{b} + \frac{r}{c} \geq \frac{3}{2}.$$

When does the equality hold?

11. Show that there is a number of the form $\underbrace{77 \dots 77}_{n\text{-sevens}}$ which is divisible by 2017.
12. Mariana decided to study for 23 days in June and to relax during the rest of the days. In how many ways can Mariana choose the seven days to relax if she decided never to relax for two days in a row?