THE MATHEMATICAL GRAMMAR SCHOOL CUP

-MATHEMATICS-

BELGRADE, June 27, 2017

PART ONE

The correct answers are: **1.** (A) **2.** (E) **3.** (B) **4.** (D) **5.** (C) **6.** (D) **7.** (A) **8.** (E)

PART TWO

9. Let k be the circumcircle of triangle ABC, and let D, E, and F be the midpoints of those arcs BC, AC, AB of k, that do not contain points A, B, C, respectively. If $P = AB \cap DF$ and $Q = AC \cap DE$, prove that PQ is parallel to BC.

Solution. Let S be the centre of the incircle of triangle ABC. It is well-known that $\{S\} = CF \cap BE \cap AD$. Firstly, we will show that point S lies on segment PQ.

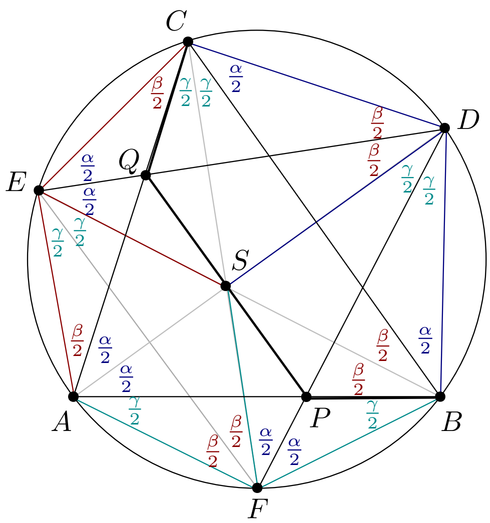
Applying the inscribed angle theorem we get that:

$$\sphericalangle ECA = \sphericalangle EAC = \sphericalangle CDE = \sphericalangle EDA = \sphericalangle AFE = \sphericalangle CFE = \frac{\beta}{2};$$
$$\sphericalangle DCB = \sphericalangle DBC = \sphericalangle DFC = \sphericalangle DFB = \sphericalangle CED = \sphericalangle BED = \frac{\alpha}{2};$$
$$\sphericalangle FAB = \sphericalangle FBA = \sphericalangle BDF = \sphericalangle ADF = \sphericalangle AEF = \sphericalangle BEF = \frac{\gamma}{2}.$$

Therefore, $\sphericalangle SAF = \sphericalangle ASF$ and $AF = SF$. Similarly, $\sphericalangle EAS = \sphericalangle ESA$, $AE = SE$ and $\sphericalangle DBS = \sphericalangle DSB$, $BD = SD$. It follows that line EF is the bisector of SA, DF is the bisector of SB, and DE is the bisector of SC.

Now, since Q lies on ED and P lies on FD, $QC = QS$ and $PS = PB$, we have that $\sphericalangle QSP = \sphericalangle QSA + \sphericalangle PSA = \sphericalangle QSE + \sphericalangle ESA + \sphericalangle ASF + \sphericalangle FSP = \beta/2 + (\alpha/2 + \beta/2) + (\alpha/2 + \gamma/2) + \gamma/2 = \alpha + \beta + \gamma = 180°$, and we conclude that Q, S, P are collinear points.

Since $\sphericalangle SPA = 180° - \sphericalangle SAP - \sphericalangle ASP = 180° - \alpha/2 - (\alpha/2 + \gamma) = \beta = \sphericalangle CBA$, we conclude that $SP \parallel BC$, i.e., that $PQ \parallel BC$. ▲



10. Let a, b, c, and m be integers and $m \geq 2$.
If $a^n + bn + c$ is divisible by m for all positive integers n, prove that b^2 is divisible by m. Does b always have to be divisible by m?

Solution. For $n = 1, 2, 3$ and some integers k_1, k_2, k_3 we have that

$$a + b + c = m \cdot k_1, \quad (1)$$
$$a^2 + 2b + c = m \cdot k_2, \quad (2)$$
$$a^3 + 3b + c = m \cdot k_3. \quad (3)$$

Now we obtain

$$(4)\qquad a^2 - a + b = m\cdot(k_2 - k_1) = m\cdot \ell_1, \qquad \text{equations (2)} - (1),$$

$$(5)\qquad a^3 - a^2 + b = m\cdot(k_3 - k_2) = m\cdot \ell_2, \qquad \text{equations (3)} - (2),$$

and therefore, from (4) and (5), $a(a-1)^2 = a^3 - 2a^2 + a = m\cdot(\ell_2 - \ell_1) = m\cdot o$.

From (4), $b = m\ell_1 - (a^2 - a)$, and from (5), $b = m\ell_2 - (a^3 - a^2)$. When we multiply these two equations we obtain

$$\begin{aligned} b^2 &= (m\ell_1 - (a^2 - a))\cdot(m\ell_2 - (a^3 - a^2)) \\ &= m(m\ell_1\ell_2 - \ell_1(a^3 - a^2) - \ell_2(a^2 - a)) + a^3(a-1)^2 \\ &= m(m\ell_1\ell_2 - \ell_1(a^3 - a^2) - \ell_2(a^2 - a)) + a^2 mo, \end{aligned}$$

which proves that b^2 is divisible by m.

Let us now show that b does not always have to be divisible by m. Let $m = 4$, $a = 3$, $b = 2$, and $c = 3$, and let us show that $3^n + 2n + 3$ is divisible by 4 for all positive integers n. We will use mathematical induction on n. When $n = 1$ we have that $4 \mid 8 = 3^1 + 2\cdot 1 + 3$. Now suppose that $3^n + 2n + 3 = 4s$, for some integer s. Let us prove that the claim holds for $n+1$: $3^{n+1} + 2(n+1) + 3 = 3\cdot(3^n + 2n + 3) - 4n - 4 = 4\cdot(3s - n - 1)$. ▲

11. If x, y, z are positive real numbers such that $x + y + z = 1$. Prove the following inequality:

$$xy + yz + zx - xyz \le \frac{8}{27}.$$

Solution 1.

$$\begin{aligned} xy + yz + zx - xyz - \frac{8}{27} &= xy(1-z) + z(x+y) - \frac{8}{27} \\ &\le (1-z)\left(\frac{x+y}{2}\right)^2 + z(x+y) - \frac{8}{27} \qquad \text{(by (AM-GM) inequality and } 1 - z \ge 0) \\ &= \frac{(1-z)^3}{4} + z(1-z) - \frac{8}{27} = \frac{\alpha^3}{4} - \alpha^2 + \alpha - \frac{8}{27} \qquad (\text{for } \alpha = 1 - z) \\ &= \left(\alpha - \frac{8}{3}\right)\left(\frac{\alpha^2}{4} - \frac{\alpha}{3} + \frac{1}{9}\right) = \left(\alpha - \frac{8}{3}\right)\left(\frac{\alpha}{2} - \frac{1}{3}\right)^2 \le 0, \end{aligned}$$

since $\alpha \in [0, 1]$, which proves the inequality. The equality holds if and only if $x = y = z = 1/3$. ▲

Solution 2. Starting from a well-known identity,

$$\begin{aligned} x^3 + y^3 + z^3 - 3xyz &= (x+y+z)(x^2 + y^2 + z^2 - xy - xz - yz) \\ &= x^2 + y^2 + z^2 - xy - xz - yz \\ &= 1 - 3(xy + xz + yz), \end{aligned}$$

we use the generalized mean inequality $M_1(x, y, z) \le M_3(x, y, z)$:

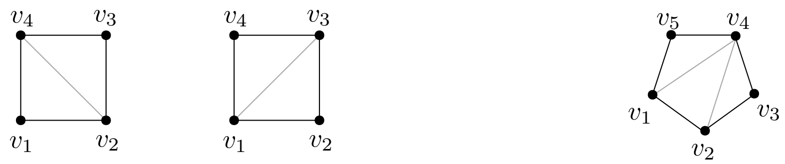
$$\sqrt[3]{\frac{x^3 + y^3 + z^3}{3}} \ge \frac{x + y + z}{3} = \frac{1}{3} \Longrightarrow x^3 + y^3 + z^3 \ge \frac{1}{9},$$

to get $1 - 3(xy + xz + yz) = x^3 + y^3 + z^3 - 3xyz \ge \frac{1}{9} - 3xyz$, and, therefore,

$$xy + yz + zx - xyz \le \frac{1}{3}\left(1 - \frac{1}{9}\right) = \frac{8}{27}.$$

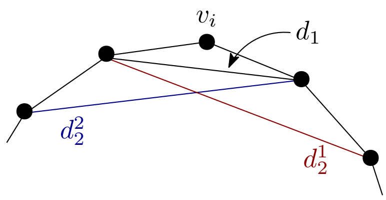
The equality holds if and only if $x = y = z = 1/3$. ▲

12. In how many ways can a convex 2017-gon be divided into triangles by 2014 diagonals that do not intersect each other (except possibly in their endpoints), in such a way that each triangle has an edge in common with that 2017-gon? The answer has to be given in the form $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, for some positive integer k, prime numbers $p_1, p_2, \ldots, p_k$, and positive integers $\alpha_1, \ldots, \alpha_k$.



Example. On the left: For a rectangle there are exactly two such divisions.
On the right: One of the divisions of a 5-gon satisfying the conditions.

Solution. Since there are 2015 triangles in this division, at least one has to have 2 sides in common with that convex 2017-gon.



Let us choose a vertex v_i (there are 2017 ways to do this). Connect the two neighbouring vertices of v_i by a diagonal d_1. The triangle whose one side is d_1, and that is on the other side of the first triangle, has to have an edge in common with the 2017-gon, so the third edge has to be one of the two diagonals connecting an endpoint of d_1 with the neighbour of the other endpoint of d_1, as shown in the figure to the right. So we have two possibilities to choose d_2. Similarly, if we have already chosen diagonals $d_1, \ldots, d_i (i < 2014)$, there are two possibilities for d_{i+1}. Therefore, we have 2017 ways to choose v_i, and 2^{2013} ways to choose a sequence $d_1, \ldots, d_{2014}$ of diagonals. However, every such division has exactly two triangles which contain two (necessarily) adjacent edges of the 2017-gon, and therefore we have to divide everything by 2 to obtain the final number of divisions with the desired property: $2017 \cdot 2^{2012}$. ▲