THE MATHEMATICAL GRAMMAR SCHOOL CUP
-MATHEMATICS-

Belgrade, June 27, 2018

The problems are divided into two parts: multiple choice questions and problems in the standard form. The students should use separate sections on the notebook provided for each of the two parts. The examination lasts 180 minutes.

PART ONE

Problems 1 to 8 are multiple choice problems. Out of the five choices offered for a problem, exactly one is the correct answer. On the answer sheet you should circle only the letter that corresponds to the answer you have chosen. Each correct answer is worth 5 points.

1. If $x > y$, $x^2 + y^2 = 5$, and $xy = \sqrt{6}$, then $x - y$ is equal to:

(A) $\sqrt{6} - 1$; (B) $\sqrt{2} - 1$; (C) $\sqrt{3} + \sqrt{2}$; (D) $\sqrt{3} - 1$; (E) $\sqrt{3} - \sqrt{2}$.

2. How many two-digit numbers n are there such that n is exactly 10 greater than three times the sum of the digits of n?

(A) 0 or 1; (B) 2; (C) 3; (D) 4; (E) 5 or more.

3. The angle between the longer base and a side of an isosceles trapezium is equal to $60°$. If the length of the longer base is 9 cm, and the side length is 4 cm, then the area of the trapezium is equal to:

(A) 18 cm^2; (B) $24\sqrt{3}$ cm^2; (C) $14\sqrt{3}$ cm^2; (D) 16 cm^2; (E) $7\sqrt{3}$ cm^2.

4. Let $ABCD$ be a square with side $a = 2$. The radius of the circle passing through midpoint E of side AB, centre O of the square, and vertex C is equal to:

(A) $\frac{1}{2}\sqrt{10}$; (B) $\frac{3}{2}$; (C) $\sqrt{3}$; (D) 2; (E) $2\sqrt{2}$.

5. The number of all 4-digit positive integers that are divisible by 5 and whose digits are distinct elements of the set $\{0, 1, 2, 5, 9\}$, is equal to:

(A) 24; (B) 48; (C) 64; (D) 84; (E) 42.

6. Let n be the smallest positive integer having remainders $2,\ 4,\ 6,\ 8$, and 10 when divided by $4,\ 6,\ 8,\ 10$, and 12, respectively. The sum of the digits of n is equal to:

(A) 9; (B) 10; (C) 11; (D) 12; (E) 13.

7. Let ABC be a triangle with $a = BC = 3$ cm and $b = AC = 4$ cm. If the sum of the lengths of the altitudes from vertices A and B, $h_a + h_b$, is equal to the length h_c of the third altitude, then $c = AB$ has length equal to:

(A) $\frac{4}{3}$ cm; (B) $\frac{20}{9}$ cm; (C) $\frac{3}{2}$ cm; (D) $\frac{12}{7}$ cm; (E) $\frac{6}{5}$ cm.

8. If $a > b > 0$, then the set of all real solutions of the inequality $ax + \frac{b}{x} < a + b$ is:

(A) $\left(-\infty, \frac{b}{a}\right) \cup (1, +\infty)$; (B) $\left(\frac{b}{a}, +\infty\right)$; (C) $\left(\frac{b}{a}, 1\right)$;

(D) $(-\infty, 0) \cup \left(\frac{b}{a}, 1\right)$; (E) $\left(0, \frac{b}{a}\right) \cup (1, +\infty)$.

PART TWO

Problems 9 to 12 are problems in the standard form, and solutions have to be written and explained in detail. Each complete solution to a problem is worth 15 points.

9. Given a regular 2018-gon, find the smallest positive integer k such that among *any* k vertices of the polygon there are 4 with the property that the convex quadrilateral they form shares 3 sides with the polygon.

10. Let the incircle of acute triangle ABC touch side BC at point D. Let us denote the points in which the incircle of triangle ABD touches sides BD, AD, and AB, by X, Y, and Z, respectively, and the points in which the incircle of triangle ACD touches sides CD and AD, by T and Y', respectively.
(a) Prove that $Y = Y'$.
(b) If lines XZ and YT intersect at point P, prove that lines PA and BC are parallel.

11. Prove that number $N = 2^{2^{2018}} - 1$ has at least 2018 distinct prime factors.

Remark: $2^{2^{2018}} = 2^{(2^{2018})}$.

12. Suppose that a, b, c are positive real numbers. Prove the following inequality:

$$\frac{a+b}{2} \cdot \frac{b+c}{2} \cdot \frac{c+a}{2} \geq \frac{a+b+c}{3} \cdot \sqrt[3]{(abc)^2}.$$