THE MATHEMATICAL GRAMMAR SCHOOL CUP
-MATHEMATICS-

Belgrade, June 28, 2016.

PART ONE

Problems 1 to 8 are multiple choice problems. Out of five offered choices for a problem, exactly one is the correct answer. On the answer sheet you should circle only the letter that corresponds to the answer you have chosen. Each correct answer is worth 5 points.

1. The set of all the solutions of the inequality $|x+3| - |x+1| < 2$ is:
(A) all real numbers; (B) $(-3, -2)$; (C) $(-\infty, -1)$; (D) $(-1, +\infty)$; (E) $(-3, -1)$.

2. The last two digits of the number $7^{9^{9^9}}$ are:
(A) 07; (B) 49; (C) 77; (D) 43; (E) 01.

3. If $x - y = 7$ and $x^2 - y^2 = 77$, then the product $x \cdot y$ is equal to:
(A) 8; (B) 18; (C) -10; (D) -15; (E) 44.

4. If $\dfrac{a+b}{b} = 3$, then $\dfrac{a^2+b^2}{ab}$ is equal to:
(A) 9; (B) $\dfrac{17}{4}$; (C) $\dfrac{10}{3}$; (D) 5; (E) $\dfrac{5}{2}$.

5. One of the bases of an isosceles trapezium has length 12 cm, while the three other sides have equal lengths, 6 cm each. The length of a diagonal of that trapezium is equal to:
(A) 6 cm; (B) $6\sqrt{2}$ cm; (C) $6\sqrt{3}$ cm; (D) 8 cm; (E) 10 cm.

6. Two chords AB and BC on circle k are equal. If the tangent line to circle k at point A forms a 50° angle with chord AB, then the acute angle between that tangent line and line BC is equal to:
(A) 30°; (B) 40°; (C) 45°; (D) 50°; (E) 60°.

7. Let x and y be any real numbers. The least possible value of the expression $x^2 + y^2 + 2x - 6y + 6$ is equal to:

(A) 6; (B) 4; (C) 0; (D) -4; (E) -6.

8. The side length of a rhombus is $a = 9$ cm, and the sum of the lengths of the diagonals of that rhombus is $d_1 + d_2 = 24$ cm. The area of the rhombus is equal to (in cm^2):

(A) 81; (B) 72; (C) 108; (D) 64; (E) 63.

PART TWO

Problems 9 to 12 are classical problems, and solutions have to be written and explained in details. Each completely correct solution of a problem is worth 15 points.

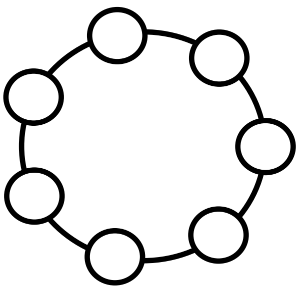
9. Let a, b, c be positive real numbers such that $a+b+c = 1$. Prove the following inequality:

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 + \left(c + \frac{1}{c}\right)^2 \geq \frac{100}{3}.$$

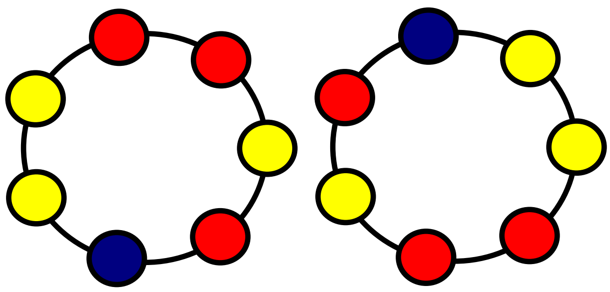
10. Let ABC be a triangle with $\sphericalangle BAC = 50^\circ$, $\sphericalangle ABC = 60^\circ$. If D and E are points on sides AB and BC, respectively, so that $\sphericalangle DCA = \sphericalangle EAC = 30^\circ$, compute the measure of angle $\sphericalangle CDE$.

11. Find all positive integers m and n so that $2^n = 3^m + 5$.

12. We are given a necklace with 7 beads, as in the figure below. Each of the beads we colour in one of the three colours (red, blue, yellow). We say that such a colouring is *colourful* if every colour is used at least once. How many different colourful colourings are there, if the colourings that could be obtained one from another by rotating the necklace are considered to be the same?



Necklace before colouring.



Two colourful colourings that are the same.