ENG

THE MATHEMATICAL GRAMMAR SCHOOL CUP

- MATHEMATICS -

26. June 2023.

PART ONE

The correct answers are: **1.** (D) **2.** (B) **3.** (C) **4.** (D) **5.** (E) **6.** (C) **7.** (A) **8.** (D)

PART TWO

9. By applying modulo 3 we see that $a \geq 2$. By applying modulo 4 we see that $b = 2l$, so difference of squares gives us:

$$2^p = c - 15^l$$
$$2^q = c + 15^l.$$

By substracting these two equations we get $2^{p-1}(2^{q-p} - 1) = 15^l$, and since 15 is odd it follows that $p = 1$, so the equation becomes:

$$2^{q-1} = 15^l + 1.$$

Suppose for a moment that $q - 1 \geq 5$. Then $32 | 15^l + 1$, but $15^l + 1$ has remainders 16 and 2 when divided by 32 and this is a contradiction, meaning $q - 1 \leq 4$. Now we easily see that the only solution is $q - 1 = 4$, i.e. $l = 1$, i.e. $a = 6$, $b = 2$. By direct check, we see that these are indeed the solution and the proof is finished.

10. With $P(x, y)$ we denote the application of the problem condition to x, y. We claim that if $3|n$ then there are exactly 2 such functions, and otherwise only one. Suppose $f(x) = f(y)$. Then from $P(x, x)$ and $P(y, y)$ we have $x = y$, i.e. f is injective. From $P(x, y)$, $P(x + y, 0)$ and injectivity we get $f(x + y) + f(0) = f(x) + f(y)$, and by easy induction we have $f(x) = ax + b$. Finally, plugging this into starting equation we have:

$$-a^2(x + y) - 2ab + b = n - x - y.$$

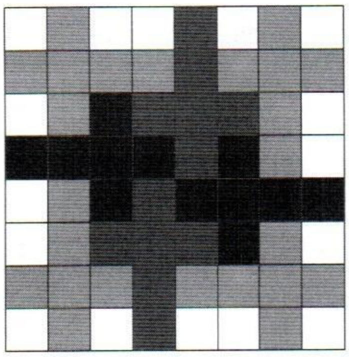
it follows that either $a = 1$ and $-b = n$ or $a = -1$ and $3b = n$. Checking that these indeed satisfy the problem statement finishes the proof.

11. We claim that on a $8 \cdot 8$ board, it is possible to place at most 8, and on a $7 \cdot 7$ board at most 5 crosses. For an example of $8 \cdot 8$ board with 8 crosses see picture, and an example for $7 \cdot 7$ is easy to construct.

We first show an upper bound for $8 \cdot 8$ board. We throw away corner squares and consider the remaining edge squares. It is obvious that no 2 covered edge squares can be adjacent, so it follows that at most $64 - 4 - 4 \cdot 3 = 48$ squares are covered, which finishes the proof.

We now show an upper bound for a $7 \cdot 7$ board. Suppose that we have placed 6 crosses without an overlap. As before, we throw away corner squares and consider the remaining edge squares. Since at most 25 non-edge squares are covered, at least $36 - 25 = 11$ edge squares are covered. Since every edge can have at most 3 covered squares (no 2 covered edge squares can be adjacent) that means that at least 3 edges have 3 squares covered, so there are 2 adjacent edges such that both have 3 squares covered. We can easily check that if 2 squares next to a corner square are covered then the middle square in one of the edges cannot be covered and we have reached a contradiction.

This discussion finishes the proof.



12. Throughout the solution, the circumcircle of XYZ will be denoted (XYZ), incircle with I, and incircle with (I). Problem condition is equivalent with $\sphericalangle FPE + \sphericalangle BPC = 180$, but then $\sphericalangle BPC = 180 - \sphericalangle FPE = 180 - \sphericalangle FDE = \sphericalangle BIC$, so the quadrilateral $BICP$ is cyclic . Because of the previous, it is enough to show that midpoints L and M, of DE and DF respectively, lie on the radical axis of (BIC) i (I), but this is easy from the power of a point because $LE \cdot LD = LI \cdot LC$ and this finishes the proof.