ENG

THE MATHEMATICAL GRAMMAR SCHOOL CUP

- MATHEMATICS -

29. June 2022.

PART ONE

The correct answers are: **1.** (C) **2.** (B) **3.** (C) **4.** (B) **5.** (E) **6.** (B) **7.** (D) **8.** (A)

PART TWO

9. For all non-negative a we have:

$$(a-1)^2(a+3) \geq 0$$
$$a^3+a^2-5a+3 \geq 0$$
$$(5-a^2)(a+1) \leq 8$$
$$\frac{1}{a+1} \geq \frac{5-a^2}{8}$$

Summed for a, b, c this gives us:

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} \geq \frac{15-a^2-b^2-c^2}{8} = \frac{3}{2}$$

10. Apply inversion with regards to A with radius $\sqrt{bc}$ and reflection in the bisector $\angle BAC$. This sends B to C, C to B, circumcircle k to the line BC and k_a to the excircle opposite of A. The inversion formula for distance states the following:

$$A_1B = A_1'B'\frac{R^2}{AA_1' \cdot AB'} = (s-b)\frac{bc}{AA_1' \cdot b}$$

Similarly we calculate A_1C and we then have:

$$\frac{A_1B}{A_1C} = \frac{(s-b)\frac{c}{AA_1'}}{(s-c)\frac{b}{AA_1'}} = \frac{(s-b)c}{(s-c)b}$$

Multiply this expression for a, b, c to get:

$$\frac{A_1B}{A_1C}\cdot\frac{B_1C}{B_1A}\cdot\frac{C_1A}{C_1B} = \frac{(s-b)c}{(s-c)b}\cdot\frac{(s-c)a}{(s-a)c}\cdot\frac{(s-a)b}{(s-b)a} = 1$$

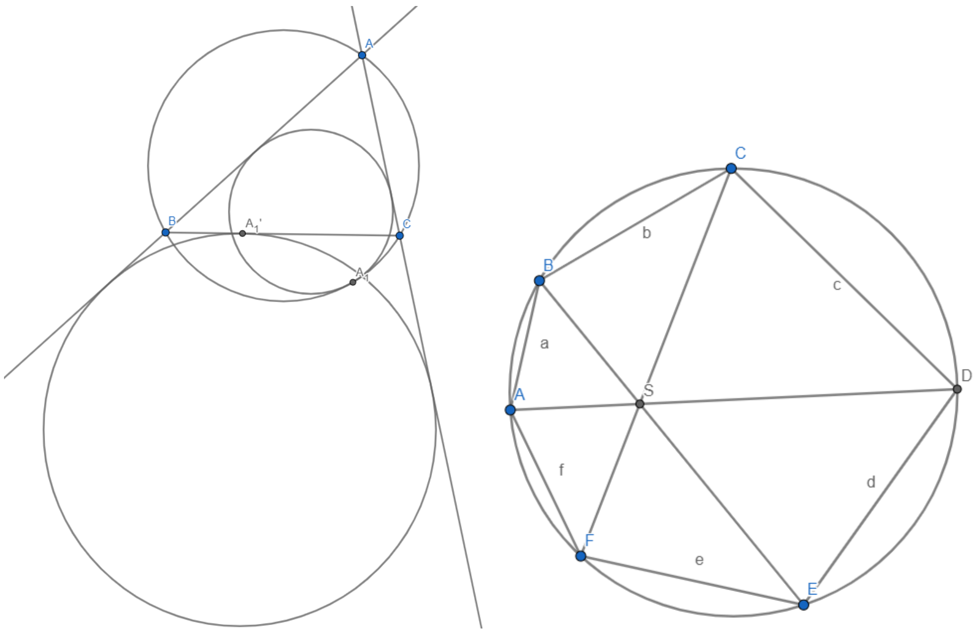
This is a well known lemma for cyclic hexagon that main diagonals intersect in a point if and only if $ace = bdf$. The proof goes as following. First we prove the direction when diagonals intersect in a point.

$$\triangle ABS \sim \triangle EDS \implies \frac{BS}{DS} = \frac{AB}{ED} = \frac{a}{d}$$
$$\triangle BCS \sim \triangle FES \implies \frac{FS}{BS} = \frac{FE}{BC} = \frac{e}{b}$$
$$\triangle CDS \sim \triangle AFS \implies \frac{DS}{FS} = \frac{CD}{AF} = \frac{c}{f}$$

Which when multiplied gives:

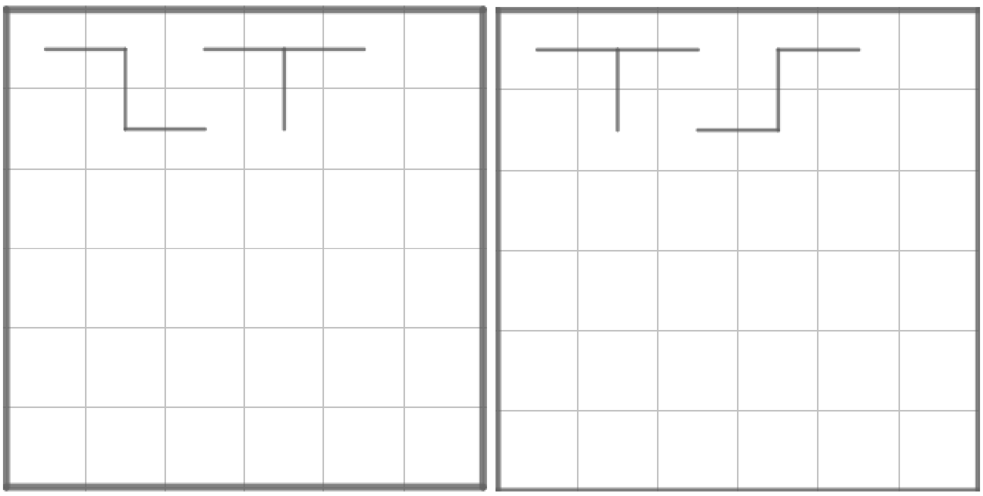
$$\frac{a}{d}\cdot\frac{e}{b}\cdot\frac{c}{f} = \frac{BS}{DS}\cdot\frac{FS}{BS}\cdot\frac{DS}{FS} = 1$$

This proves one direction of the lemma. Other other is proven using the uniqueness argument. Suppose the points A, B, C, D, E are fixed. If S is intersection of BE and AD then there is a unique F such that CF contains S. For this F we have $\frac{ace}{bdf} = 1$. However there is a unique F on the arc EA such that $\frac{f}{e} = \frac{ac}{bd}$. Hence if $\frac{ace}{bdf} = 1$ holds, this has to be the F for which AD, BE and CF intersect in a point. Hence the other direction of the lemma is proven.



(A) k_a and the excircle (B) Hexagon lemma

11. We shall prove that a 6×6 board cannot be tiled. Suppose a corner square is covered by a skew-tetromino. Then the tetromino next to it has to be a T-tetromino:



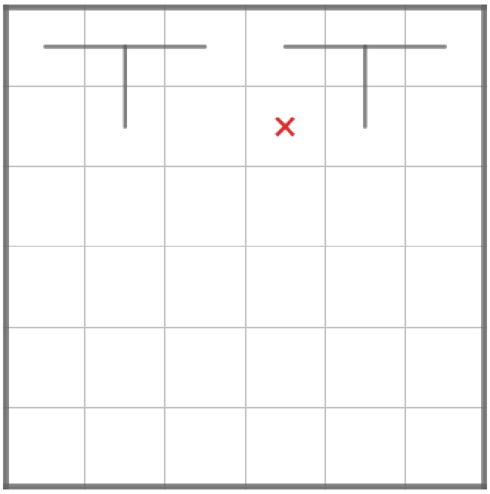
(A) corner covered by skew (B) equivalent covering

Then we notice we can cover the corner equivalently with a T-tetromino. Hence we can assume w.l.o.g. that all corners are covered by T-tetromino. If two corners are covered T-tetrominos oriented towards each other, notice that the red tile cannot be covered without causing another tile to become uncoverable. We're left with the case where T-tetrominos all oriented differently, but note that then the square in center cannot be covered.

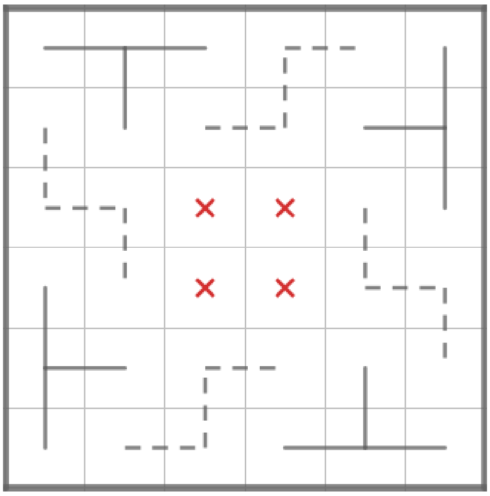
Hence a 6×6 board cannot be tiled. A 10×10 board can be covered as in the picture below.

To cover a 2022×2022 board we shall require covering 4×4 and 4×6 boards:

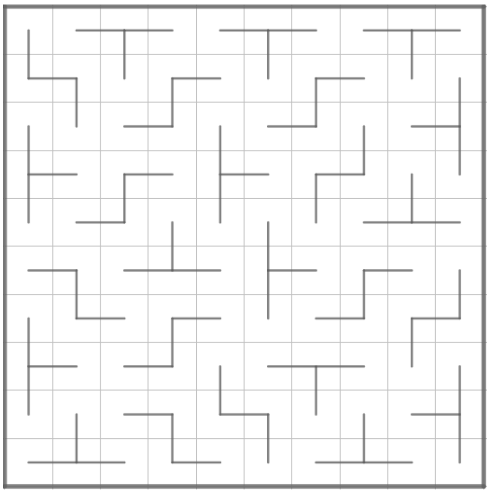
Next we divide the 2022×2022 board into 10×10, two 10×2012 and 2012×2012 boards. Further 10×2012 can be divided into 10×4 boards that separate into 4×4 and 4×6 boards. The 2012×2012 board can be divided into 4×4 boards. Each of the boards pieces can now be covered.



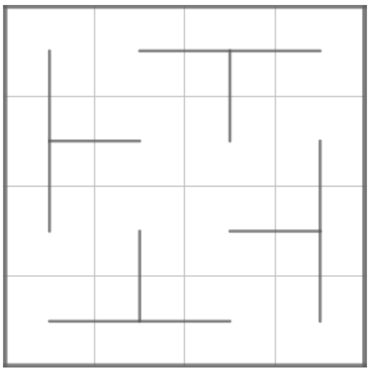
(A) two T corners



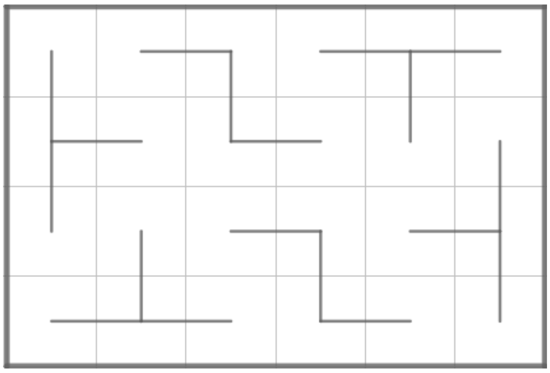
(B) best covering



(A) 10×10 covering



(A) 4×4 covering



(B) 4×6 covering

12. Modulo 4 gives us:

$$9^n + 10^n + 11^n \equiv 1 + 3^n \equiv 0, 1 (mod 4)$$

for $n/ge2$. Hence n is odd. Next modulo 9 gives us:

$$9^n + 10^n + 11^n \equiv 1 + 2^n \equiv 0, 1, 4, 7 (mod 9)$$

This gives us that $n = 6k + 3$ Next we observe modulo 13:

$$9^n + 10^n + 11^n \equiv (-4)^n + (-3)^n + (-2)^n \equiv -64 - 27 - 8(-1)^k$$

$$\equiv 8(-1)^{(k+1)} \equiv 0, 1, 3, 4, 9, 10, 12 (mod 13)$$

The expression can only give residues 5 or 8 modulo 13. However neither are quadratic residues. Hence there exists no n for which $9^n + 10^n + 11^n$ is square.