The Mathematical Grammar School Cup

≈ MATHEMATICS SOLUTIONS ≈

PART ONE

1. B) 14	**2.** D) 336	**3.** F) 9	**4.** D) 214
5. C) $2 \cdot 3^{2021}$	**6.** E) 5	**7.** E) $\pi - 1, 92$	**8.** C) 13924

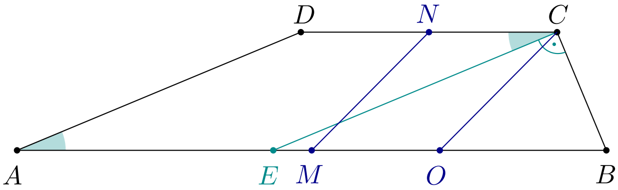
PART TWO

9. Let $ABCD$ be a trapezoid such that $AB \parallel CD$ and $AB = 23$, $BC = 5$, $CD = 10$. Furthermore, $\sphericalangle DCB = 90^\circ + \sphericalangle BAD$.

(a) Compute the length of side AD.

(b) Let M and N be the midpoints of segments AB and CD, respectively. Compute the length of segment MN.

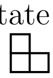
(c) Compute the length h of the altitude of this trapezoid.



Solution. Let E be the point on side AB such that $\sphericalangle BCE = 90^\circ$. Since $\sphericalangle BCE + \sphericalangle ECD = \sphericalangle DCB = 90^\circ + \sphericalangle BAD$, we conclude that $\sphericalangle ECD = \sphericalangle BAD$. Hence, rectangle $AECD$ is a parallelogram and $AD = EC$, $AE = 10$. Triangle EBC is right angled at C, with $CB = 5$ and $EB = 23 - 10 = 13$. Therefore, $EC = 12$ ($(5, 12, 13)$ is a well-known Pythagorean triple), and $AD = 12$.

Let O be the the point of EB such that $MOCN$ is a parallelogram. Then $BO = BM - MO = \frac{23}{2} - \frac{10}{2} = \frac{13}{2}$, i.e., point O is the midpoint of hypotenuse BE of triangle EBC. Therefore, $CO = BO = EO$, and $MN = \dfrac{13}{2}$.

Finally, we compute the area of $\triangle EBC$ in two different ways to obtain $\dfrac{h \cdot EB}{2} = \dfrac{CE \cdot CB}{2}$ and, hence, $h = \dfrac{60}{13}$.

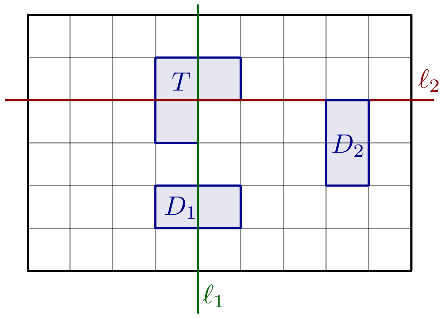
10. A *tiling* of a board is a way to place several tiles on that board so that all of its squares are covered, but no tiles overlap or lie partially off the board. It is allowed to rotate tiles. In this problem we will consider tilings by *dominoes* () or L–shaped *trominoes* ().

(a) How many different tilings of a 2×10 board by dominoes are there?

(b) How many different tilings of a 3×100 board by trominoes are there?

(c) Let $n \times m$, for $2 \leqslant n \leqslant m$, be the board of the smallest area which can be tiled by dominoes in such a way that every line parallel to the board sides, that intersects the interior of the board, also intersects the interior of *at least one domino*. What are n and m equal to?

(d) Draw at least one tiling of a 4×9 board by trominoes in such a way that that every line parallel to the board sides, that intersects the interior of the board, also intersects the interior of *at least one tromino*.

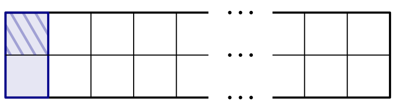


For example, in the picture above line ℓ_1 intersects the interior of both domino D_1 and tromino T, while ℓ_2 intersects only the interior of tromino T.

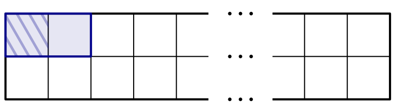
Solution.

(a) Denote by a_n the number of different tilings of a $2 \times n$ board by dominoes. Our task is to compute a_{10}.

Let us take a look at the top–left corner of the board. It can be covered by a domino in one of the two ways shown in the picture to the right.



(i)



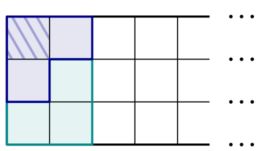
(ii)

In the case (i), we are left with tilings of a $2 \times (n-1)$ board, and there are a_{n-1} of those. In the case (ii), a horizontal domino must be placed below the one covering top-left corner, and we are again left with tilings of a smaller board – this time it is a $2 \times (n-2)$ board with a_{n-2} tilings.

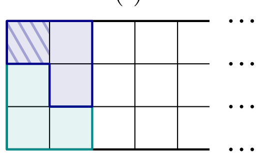
We conclude that $a_n = a_{n-1} + a_{n-2}$. So, in order to compute a_{10}, we need to compute a_1 and a_2, which is very easy. Namely, $a_1 = 1$ (one vertical domino) and $a_2 = 2$ (both dominoes are either horizontal or vertical). Now we have $a_3 = 3$, $a_4 = 5$, $a_5 = 8$, $a_6 = 13$, $a_7 = 21$, $a_8 = 34$, $a_9 = 55$, and finally $a_{10} = 89$.

Comment: Some students might notice that, for $n \geqslant 1$, $a_n = f_{n+1}$, where (f_n) is the famous *Fibonacci sequence*.

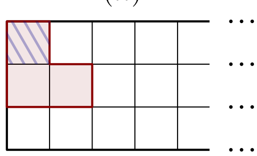
(b) We proceed with the same idea as in the part (a), by looking at the top–left corner of the board, see the picture to the right.



(i)



(ii)

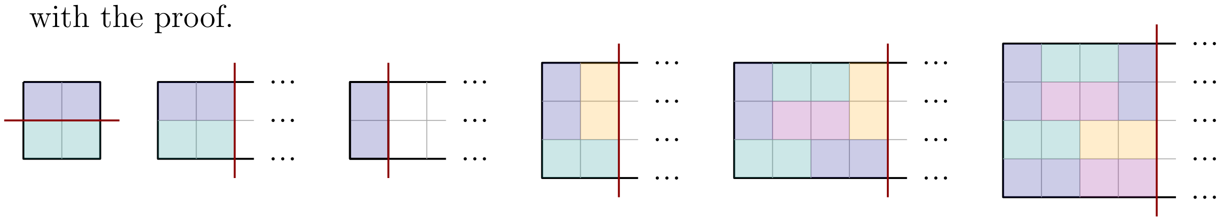


(iii)

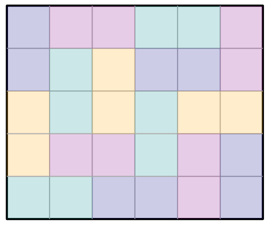
In both cases (i) and (ii) there is a unique way to put a tromino to cover down–left corner of the board, while the situation depicted in (iii) is not possible since it does not yield a valid tiling.

Therefore, if b_n is the number of different tilings of a $3 \times n$ board by trominoes, we have $b_n = 2b_{n-2}$, for $n \geqslant 3$, and therefore $b_{100} = 2^{49} \cdot b_2$. Having in mind that $b_2 = 2$, we obtain the desired number, $b_{100} = 2^{50}$.

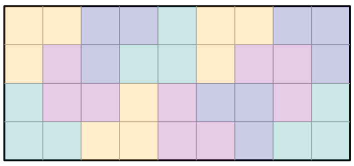
(c) It can be checked that, if $n \in \{2, 3, 4\}$, there always exists a line parallel to the board sides, that intersects the interior of the board, but does not intersect the interior of any domino – see the picture below for some cases, which illustrate how one could proceed with the proof.



Therefore, we must try to construct a required tiling for $n = 5$. The smallest $m \geqslant n$ for which any tiling by dominoes exists is $m = 6$, and we claim that $n = 5$ and $m = 6$ is our answer, see the picture on the right.



(d) One such tiling is given in the picture below.



11. Let $S(n)$ be the sum of digits of a positive integer n.

(a) Find the smallest positive integer n such that $9 \mid S(n)$ and $9 \mid S(n+1)$. If no such number exists, write 'X'.

(b) Find the smallest positive integer n such that $11 \mid S(n)$ and $11 \mid S(n+1)$. If no such number exists, write 'X'.

(c) Finally, find the number of positive integers n with at most 8 digits for which it holds that $11 \mid n$, $11 \mid S(n)$, and $11 \mid S(n+1)$, and write all of them.

Solution.

(a) Suppose that, for some positive integer n, we have $9 \mid S(n)$ and $9 \mid S(n+1)$. From the fact that $n \equiv S(n) \pmod 9$ it follows that $0 \equiv S(n+1) - S(n) \equiv (n+1) - n = 1 \pmod 9$, which is a contradiction. Therefore the answer is X, there does not exist such a number.

(b) Firstly, note that if the last digit of a positive integer n is not equal to 9, then $S(n+1) - S(n) = 1$, and the difference cannot be divisible by 11. Therefore, if a number n such that $11 \mid S(n)$ and $11 \mid S(n+1)$ exists, its last k digits, for some $k \geqslant 1$, must be equal to 9:

$$n = \overline{a_m \ldots a_k \underbrace{9 \ldots 9}_{k}}, \text{ with } a_k \neq 9.$$

For such n, $S(n) = a_m + \cdots + a_k + 9k$ and $S(n+1) = a_m + \cdots + (a_k + 1)$. Therefore, $11 \mid S(n) - S(n+1) = 9k - 1$ if and only if $k = 5 + 11s$, for a nenegative integer s. We seek the smallest positive integer n satisfying conditions in (b) among numbers ending with five nines. That number cannot be a six–digit number $\overline{a99999}$ because $11 \nmid a + 1 = S(n+1)$, for $a \in \{1, 2, 3, \dots, 8\}$. Let us proceed with 7–digit numbers $\overline{ab99999}$ such that $b \neq 9$ and $11 \mid a + b + 1 = S(n+1)$. Since $b \leqslant 8$, we get that $a \geqslant 2$, and therefore, the smallest positive integer n satisfying $11 \mid S(n)$ and $11 \mid S(n+1)$ is $n = 2899999$.

(c) From the part (b), we know that such a number n must have at least 7 digits and must end with five nines. Firstly, suppose that $n = \overline{ab99999}$, for $b \leqslant 8$. Then (part (b)), $11 \mid S(n)$ and $11 \mid S(n+1)$ if and only if $11 \mid a+b+1$. Since $11 \mid n$, we get that $11 \mid a+3\cdot 9-(b+2\cdot 9)$, i.e., $11 \mid a-b+9$. It is now easy to see that the only such 7–digit number is $n = 6499999$.

Now, let $11 \mid S(n)$, $11 \mid S(n+1)$, $11 \mid n$, and $n = \overline{abc99999}$, $a \geqslant 1$, $c \neq 9$. Part (b) gives us that the first two conditions are met if and only if $11 \mid a+b+c+1$ $(*)$, while $11 \mid n$ if and only if $11 \mid a+c-b-9$ $(**)$. By substracting $(**)$ from $(*)$ we get that $11 \mid 2b+10$, so b must be equal to 6. Hence, $11 \mid a+c+7$, and $(a, c) \in \{(1,3), (2,2), (3,1), (4,0), (7,8), (8,7), (9,6)\}$. Therefore, the answer to part (c) is: there are eight such numbers, and they are 6499999, 16399999, 26299999, 36199999, 46099999, 76899999, 86799999, 96699999 (it is easy to check that for all these numbers all conditions given in the part (c) are met).

12. (a) Find the minimum m_a of the expression $\dfrac{x^2+1}{x}$, for $x > 0$.

(b) Find the minimum m_b of the expression $\dfrac{x^3+3x+9}{x^2}$, for $x > 0$.

(c) Find the minimum m_c of the expression $\dfrac{(x+1)(y+2)(xy+2)}{xy}$, for $x > 0$ and $y > 0$.

(d) Find the minimum m_d of the expression $\dfrac{(x+4)(y+1)(xy+864)}{xy}$, for $x > 0$ and $y > 0$.

Solution.

(a) Using the Arithmetic–Geometric Mean Inequality (AG) we get $x + \dfrac{1}{x} \geqslant 2\sqrt{x \cdot \dfrac{1}{x}} = 2$. The equality holds for $x = 1$, and $m_a = 2$.

(b) $\dfrac{x^3+3x+9}{x^2} = \dfrac{\frac{x^3}{3}+\frac{x^3}{3}+\frac{x^3}{3}+3x+9}{x^2} \overset{\text{AG}}{\geqslant} \dfrac{1}{x^2}\cdot 5 \cdot \sqrt[5]{\dfrac{x^3}{3}\cdot\dfrac{x^3}{3}\cdot\dfrac{x^3}{3}\cdot 3x \cdot 9} = 5$. The equality holds if and only if $\frac{x^3}{3} = 3x = 9$, i.e., iff $x = 3$. Therefore, $m_b = 5$.

(c) $\dfrac{(x+1)(y+2)(xy+2)}{xy} \overset{\text{AG}\times 3}{\geqslant} \dfrac{2\sqrt{x}\cdot 2\sqrt{2y}\cdot 2\sqrt{2xy}}{xy} = 16$. The equality holds if and only if $x = 1$, $y = 2$, and $xy = 2$. Hence, $m_c = 16$.

(d) Firstly, let us prove that, for $x, y, z > 0$, $(x^3+1)(y^3+1)(z^3+1) \geqslant (xyz+1)^3$ (I). This inequality is equivalent with $x^3y^3 + y^3z^3 + z^3x^3 + x^3 + y^3 + z^3 \geqslant 3x^2y^2z^2 + 3xyz$, and this is true since

$$x^3y^3 + y^3z^3 + z^3x^3 \overset{\text{AG}}{\geqslant} 3\sqrt[3]{x^6y^6z^6} = 3x^2y^2z^2 \text{ and } x^3 + y^3 + y^3 \overset{\text{AG}}{\geqslant} 3\sqrt[3]{x^3y^3z^3} = 3xyz.$$

We see that the equality holds if and only if $x = y = z$. Now we use this to get:

$$\frac{(x+4)(y+1)(xy+864)}{xy} = 4\left(\frac{x}{4}+1\right)(y+1)\left(\frac{864}{xy}+1\right) \overset{\text{(I)}}{\geqslant} 4\left(\sqrt[3]{\frac{x}{4}\cdot y \cdot \frac{864}{xy}}+1\right)^3 = 1372.$$

The equality holds if and only if $\frac{x}{4} = y = \frac{864}{xy}$, i.e., for $x = 24$ and $y = 6$. Hence, $m_d = 1372$.