THE MATHEMATICAL GRAMMAR SCHOOL CUP

–MATHEMATICS SOLUTIONS–

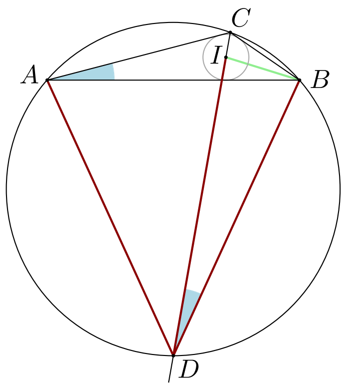
BELGRADE, June 23, 2020

PART ONE

The correct answers are: **1.** (A) **2.** (B) **3.** (A) **4.** (C) **5.** (E) **6.** (B) **7.** (D) **8.** (E)

PART TWO

9. Let ABC be a triangle with $BC = 4$ cm, $CA = 9$ cm, $AB = 12$ cm, and let I be the incenter of $\triangle ABC$. If D is the other intersection of CI with the circumcircle of $\triangle ABC$, and if $CD = 15,6$ cm, compute the length of CI.



Solution. It is well known that bisector CI intersects the circumcircle of $\triangle ABC$ at the middle of arc AB, and therefore $DA = DB$. Since $\sphericalangle IBD = \sphericalangle IBA + \sphericalangle ABD = \beta/2 + \gamma/2$, $\sphericalangle IDB = \alpha$, it follows that $\sphericalangle BID = 180° - \sphericalangle IBD - \sphericalangle IDB = \alpha + \beta + \gamma - \beta/2 - \gamma/2 - \alpha = \beta/2 + \gamma/2 = \sphericalangle IBD$. In other words, triangle BID is isosceles, and therefore $DA = DI = DB = x$.
Now we can use Ptolomy's theorem for the cyclic quadrilateral $ADBC$ to obtain

$$AB \cdot CD = AC \cdot BD + AD \cdot BC, \text{ i.e.,}$$
$$12 \cdot 15,6 = 9 \cdot x + x \cdot 4.$$

Therefore, $x = 14,4$ cm, and $CI = 1,2$ cm. ▲

10. Let $f(n)$ be a function that counts the number of positive integers bigger than n and not larger than $n + 100$ that are divisible by 3 or 7, but not by both. Let M be the maximal value of the function f on the set $\{1, 2, 3, \ldots, 2020\}$ and let k be the biggest positive integer not greater than 2020 such that $f(k) = M$.

(a) Find M.
(b) Find $k + f(k)$.

Solution. Since, for every n, there are exactly 100 positive integers k such that $n < k \le n + 100$, among those there are at most $\lceil \frac{100}{3} \rceil$ numbers divisible by 3, at most $\lceil \frac{100}{7} \rceil$ divisible by 7, and at least $\lfloor \frac{100}{21} \rfloor$ divisible by 21. Therefore,

$$f(n) \le \left\lceil \frac{100}{3} \right\rceil + \left\lceil \frac{100}{7} \right\rceil - 2 \cdot \left\lfloor \frac{100}{21} \right\rfloor = 41.$$

If we assume that there is a number n which satisfies the equality $f(n) = 41$, then all the equalities above would have to hold, i.e., there would have to be 34 numbers divisible by 3, 15 divisible by 7 and only 4 divisible by 21.

Firstly, we consider divisibility by 3. If one of the endpoints $n + 1$ and $n + 100$ is divisible by 3, then both of them will be, since their difference is 99. This is the only possibility that allows to have 34 numbers divisible by 3, as otherwise, there is at most $\lceil \frac{99}{3} \rceil$ of them. Therefore, we need to have $n \equiv 2 \pmod 3$. Similarly, considering divisibility by 7, we that the only way to have 15 numbers divisible by 7 is when (exactly) one of the endpoints $n + 1$ and $n + 100$ is divisible by 7. These are the cases $n \equiv 5, 6 \pmod 7$. However, in both of these cases one of the endpoints is divisible by 21, and hence there are 5 numbers divisible by 21 in the set $\{n + 1, n + 2, \ldots, n + 100\}$.

We have proved that $f(n) \le 40$. It is not too strenuous to check that $f(2) = 40$, and hence $M = 40$.

It is pretty straightforward to see that $f(2020) = f(2019) = 39$ and that $f(2018) = 40$, and therefore $k = 2018$. Finally $k + f(k) = k + M = 2058$. ▲

11. (a) Find some numbers $b, c \in \mathbb{R}$ so that $|x^2 + bx + c| \le \frac{1}{2}$ holds for all $x \in [-1, 1]$.
(b) Find some numbers $b, c, e, f \in \mathbb{R}$ so that $|x^2+bxy+y^2+dx+ey+f| \le 1$ holds for all $x, y \in [-1, 1]$.
(c) Find some numbers $b, c, e, f \in \mathbb{R}$ so that $|x^2+bxy+y^2+dx+ey+f| \le \frac{1}{2}$ holds for all $x, y \in [0, 1]$.
(d) Find some numbers $a, b, c, d, e, f, g, h, i, j \in \mathbb{R}$ so that

$$|x^2y^2 + ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j| \le \frac{1}{4}$$

holds for all $x, y \in [-1, 1]$.

Solution.

(a) Substituting $x = 0, 1, -1$ we obtain the following inequalities

$$(1) \qquad |c| \le \frac{1}{2} \iff -\frac{1}{2} \le c \le \frac{1}{2},$$

$$(2) \qquad |1+b+c| \le \frac{1}{2} \iff -\frac{1}{2} \le 1+b+c \le \frac{1}{2},$$

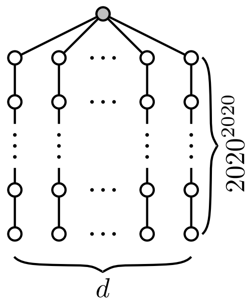
$$(3) \qquad |1-b+c| \le \frac{1}{2} \iff -\frac{1}{2} \le 1-b+c \le \frac{1}{2}.$$

By adding inequalities (2) and (3), we obtain that $-1 \le 2 + 2c \le 1$, i.e., $-\frac{3}{2} \le c \le -\frac{1}{2}$. Combining this and (1), we get that c must be equal to $-\frac{1}{2}$, and substituting this into (2) and (3), we get $-1 \le b \le 0$ and $0 \le b \le 1$, i.e., $b = 0$. Now it is easy to check that $|x^2 - \frac{1}{2}| \le \frac{1}{2}$ for all $x \in [-1, 1]$. This part has a unique solution $b = 0,\ c = -\frac{1}{2}$.

(b) $x^2+y^2-1 = (x^2-\frac{1}{2})+(y^2-\frac{1}{2})$ satisfies the conditions because of part (a): $b = d = e = 0,\ f = -1$.

(c) One such polynomial is $x^2 - 2xy + y^2 - \frac{1}{2} = (x-y)^2 - \frac{1}{2}$, $b = -2,\ d = e = 0,\ f = -\frac{1}{2}$.

(d) One such polynomial is $(x^2 - \frac{1}{2})(y^2 - \frac{1}{2}) = x^2y^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{1}{4}$, $a = b = c = d = f = h = i = 0$, $e = g = -\frac{1}{2}$, $j = \frac{1}{4}$. ▲



12. Let d be a positive integer. In a country called Matgimia there are $2020^{2020} \cdot d + 1$ towns that are connected by two-way roads in the following way: the capital of the country is linked to exactly d other towns, while all other towns are organized in d chains of length 2020^{2020} which start from the capital, as shown in the figure. Let $c(X)$ be the number of roads leaving town X ($c(X) = d$ if X is the capital, $c(X) = 1$ for the towns that are at ends of the chains, and $c(X) = 2$ otherwise). On January 1st, n tourists arrived in the capital of Matgimia. On each of the following days, for each town X in which there are at least $c(X)$ tourists, exactly $c(X)$ tourists leave that town and go to the neighboring towns, each one using a different road. After a while, it turned out that all the tourists stopped traveling and that there is at most one tourist in each of the towns. Let $n_1 < n_2 < n_3 < \cdots$ be all positive integers such that this situation is possible for $n = n_k$, $k = 1, 2, \ldots$. Find n_{2020} if

(a) $d = 1$; (b) $d = 2$; (c) $d = 3$; (d) $d = 2020$.

Solution. The answers are: (a) 2020, (b) 2020, (c) 3030, (d) $2020 \cdot 1010 = 2040200$.

We can see that for any n, the number of tourists in the final configuration that stay in the capital, denote it by ℓ, is at most $d - 1$. In all other towns the corresponding number is at most one. Since all the chains are in symmetrical positions, the number of tourists in each of the chains is equal; denote this number by k. Now we have that $n = kd + \ell$, and since $\ell \le d - 1$, we have that k and ℓ are uniquely determined (dividing n by d, there are unique quotient and remainder). Since it must be that $\ell \le 1$, we get that $n = kd$ or $n = kd + 1$. Since the length of chains is very large, one can prove that, for $n_{2k+1} = kd + 1$, $k \in \{0, 1, 2, \ldots, 1010\}$ and for $n_{2k} = kd$, $k \in \{1, 2, \ldots, 1010\}$, the process described above will indeed end and it will end in the desired configuration. ▲