THE MATHEMATICAL GRAMMAR SCHOOL CUP

-MATHEMATICS-

BELGRADE, June 27, 2018

PART ONE

The correct answers are: **1.** (E) **2.** (B) **3.** (C) **4.** (A) **5.** (E) **6.** (B) **7.** (D) **8.** (D)

PART TWO

9. Given a regular 2018-gon, find the smallest positive integer k such that among *any* k vertices of the polygon there are 4 with the property that the convex quadrilateral they form shares 3 sides with the polygon.

Solution. Number the vertices of the given polygon from 1 to 2018. What we want is that any set of k vertices contains at least 4 consecutive ones.

Consider the 4-tuples of consecutive vertices,

$$(1,2,3,4),\ (2,3,4,5),\ldots,\ (2017,2018,1,2),\ (2018,1,2,3).$$

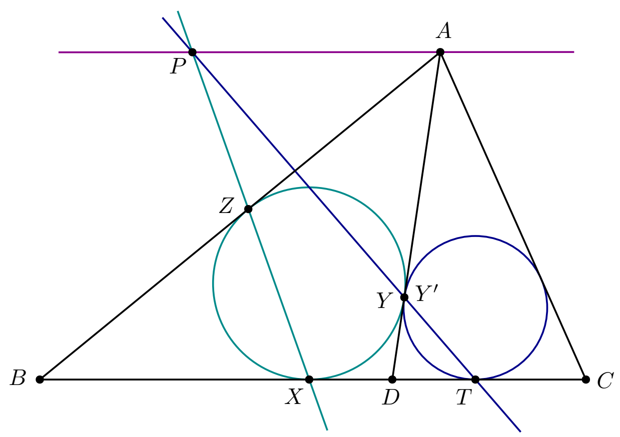
There are 2018 different ones. Each vertex uses exactly four of these 4-tuples.

Therefore, if $4k > 3 \cdot 2018$, by the Pigeon-hole principle, there is a 4-tuple with four of the k vertices. That is, if $k \geq 1514$, the condition we want is met.

To see that 1514 is the minimum, it is enough to take all the vertices, except the multiples of 4 and 2018. That way we get a set of $2018 - 2016/4 - 1 = 2018 - 505 = 1513$ vertices with no four consecutive ones. ▲

10. Let the incircle of acute triangle ABC touch side BC at point D. Let us denote the points in which the incircle of triangle ABD touches sides BD, AD, and AB, by X, Y, and Z, respectively, and the points in which the incircle of triangle ACD touches sides CD and AD, by T and Y', respectively.
(a) Prove that $Y = Y'$.
(b) If lines XZ and YT intersect at point P, prove that lines PA and BC are parallel.

Solution.



(a) Since D is the touching point of the incircle of $\triangle ABC$ with BC, we have that

$$CD = \frac{1}{2}(AC + BC - AB),\ BD = \frac{1}{2}(AB + BC - AC).$$

Now we use the fact that Y and Y' are also the touching points of the corresponding incircles to obtain:

$$AY = \frac{1}{2}(AB + AD - BD), \quad AY' = \frac{1}{2}(AC + AD - CD).$$

From these equalities we obtain that

$$\begin{aligned} AY - AY' &= \frac{1}{2}(AB - AC + CD - BD) \\ &= \frac{1}{2}\left(AB - AC + \frac{1}{2}(AC + BC - AB) - \frac{1}{2}(AB + BC - AC)\right) = 0, \end{aligned}$$

and therefore $Y = Y'$.

(b) Let p be the line parallel to BC containing point A. It is our goal to show that $p = AP$. Let $\{P_1\} = p \cap XZ$, $\{P_2\} = p \cap YT$.

Since $p \parallel BC$ and $\sphericalangle AZP_1 = \sphericalangle XZB$, we can conclude that $\triangle AP_1Z \sim \triangle BXZ$. Since $BX = BZ$, it follows that $AP_1 = AZ$.

Similarly, we can conclude that $AP_2 = AY$, and, since $AZ = AY$, it follows that $AP_1 = AP_2$. Hence, $P_1 \equiv P_2 \equiv P$, and therefore, $p \equiv AP$. ▲

11. Prove that number $N = 2^{2^{2018}} - 1$ has at least 2018 distinct prime factors.

Solution. First notice that

$$\begin{aligned} 2^{2^{2018}} - 1 &= \left(2^{2^{2017}} - 1\right) \cdot \left(2^{2^{2017}} + 1\right) = \left(2^{2^{2016}} - 1\right) \cdot \left(2^{2^{2016}} + 1\right) \cdot \left(2^{2^{2017}} + 1\right) \\ &= \cdots = \underbrace{\left(2^{2^0} - 1\right)}_{=1} \cdot \left(2^{2^0} + 1\right) \cdot \ldots \cdot \left(2^{2^{2016}} + 1\right) \cdot \left(2^{2^{2017}} + 1\right) \end{aligned} \tag{1}$$

Since there are 2018 factors in the product on the right hand side of the previous equality, it suffices to show that all these factors are relatively prime in pairs.

Let $F_k = 2^{2^k} + 1$. Let us show that $(F_k, F_j) = 1$, for all $1 \leq j < k$. Analogously to (1), we obtain that

$$F_k - 2 = F_{k-1} \cdot F_{k-2} \cdot \ldots \cdot F_1 \cdot F_0. \tag{2}$$

Now suppose that, for some $1 \leq j < k$, $d \mid F_j$ and $d \mid F_k$, where $d \in \mathbb{N}$. Using (2), we get that $d \mid 2$. Since F_i is odd, for $i \geq 1$, we conclude that d must be equal to 1. In other words, $(F_k, F_j) = 1$.

Now, let p_i be any prime factor of F_i. Since $2^{2^{2018}} - 1 = F_{2017} \cdot F_{2016} \cdot \ldots \cdot F_0$, we conclude that $p_i \mid 2^{2^{2018}} - 1$, for all $i \in \{0, 1, \ldots, 2017\}$. Finally, we get that $p_0 p_1 \cdots p_{2017} \mid 2^{2^{2018}} - 1$, and since $p_i \neq p_j$, for $i \neq j$, we have proved that $2^{2^{2018}} - 1$ has at least 2018 distinct prime factors. ▲

12. Suppose that a, b, c are positive real numbers. Prove the following inequality:

$$\frac{a+b}{2} \cdot \frac{b+c}{2} \cdot \frac{c+a}{2} \geq \frac{a+b+c}{3} \cdot \sqrt[3]{(abc)^2}. \tag{3}$$

Solution 1. Let

$x = \dfrac{a}{a+b+c}$, $y = \dfrac{b}{a+b+c}$, $z = \dfrac{c}{a+b+c}$, and note that $x + y + z = 1$, $x, y, z > 0$.

With these notations, the inequality (3) is equivalent with

$$\frac{x+y}{2} \cdot \frac{y+z}{2} \cdot \frac{z+x}{2} \geq \frac{1}{3} \cdot \sqrt[3]{(xyz)^2}. \tag{4}$$

Now we have that

$$\frac{x+y}{2}\cdot\frac{y+z}{2}\cdot\frac{z+x}{2}=\frac{1-z}{2}\cdot\frac{1-x}{2}\cdot\frac{1-y}{2}=\frac{1}{8}\left(1-(x+y+z)+(xy+yz+zx)-xyz\right)$$
$$=\frac{1}{8}\left(xy+yz+zx-xyz\right)=\frac{xyz}{8}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right),$$

i.e., the inequality (4) is equivalent with

$$\sqrt[3]{xyz}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)\geq\frac{8}{3}. \tag{5}$$

The Harmonic Mean–Geometric Mean–Arithmetic Mean Inequality gives us that

$$\frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}\leq\sqrt[3]{xyz}\leq\frac{x+y+z}{3}=\frac{1}{3},\text{ and therefore}$$
$$\sqrt[3]{xyz}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)=\sqrt[3]{xyz}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-\sqrt[3]{xyz}\geq 3-\sqrt[3]{xyz}\geq 3-\frac{1}{3}=\frac{8}{3},$$

which is exactly what we were supposed to show.

The equality holds if and only if $x=y=z$, i.e., if and only if $a=b=c$. ▲

Solution 2. First, we will show that the inequality

$$(a+b)(b+c)(c+a)\geq\frac{8}{9}(a+b+c)(ab+bc+ca) \tag{6}$$

holds for all positive real numbers a,b,c. Namely,

$$(6)\iff 2abc+a^2b+ab^2+a^2c+ac^2+b^2c+bc^2\geq\frac{8}{9}\left(3abc+a^2b+ab^2+a^2c+ac^2+b^2c+bc^2\right)$$
$$\iff a^2b+ab^2+a^2c+ac^2+b^2c+bc^2-6abc\geq 0.$$

Using the Arithmetic Mean–Geometric Mean Inequality, we obtain

$$\frac{1}{6}\left(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2\right)\geq\sqrt[6]{a^2\cdot b\cdot a\cdot b^2\cdot a^2\cdot c\cdot a\cdot c^2\cdot b^2\cdot c\cdot b\cdot c^2}=abc,$$

which shows that inequality (6) is satisfied.

Finally, using (6), we get that

$$(a+b)(b+c)(c+a)\geq\frac{8}{9}(a+b+c)(ab+bc+ca)=\frac{8}{3}(a+b+c)\frac{ab+bc+ca}{3}$$
$$\geq\frac{8}{3}(a+b+c)\cdot\sqrt[3]{ab\cdot bc\cdot ca}=\frac{8}{3}(a+b+c)\cdot\sqrt[3]{a^2b^2c^2},$$

which completes our proof. Note that the equality holds if and only if $a=b=c$. ▲