THE MATHEMATICAL GRAMMAR SCHOOL CUP

-MATHEMATICS-

BELGRADE, June 28, 2016

PART ONE

The correct answers are: **1.** (C) **2.** (A) **3.** (B) **4.** (E) **5.** (C) **6.** (A) **7.** (D) **8.** (E)

PART TWO

9. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove the following inequality:

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2 \geq \frac{100}{3}.$$

Solution. First we use the inequality between the quadratic and the arithmetic mean for numbers $a_1 = a+\frac{1}{a}$, $b_1 = b+\frac{1}{b}$, and $c_1 = c+\frac{1}{c}$, which are all positive:

$$\sqrt{\frac{a_1^2+b_1^2+c_1^2}{3}} \geq \frac{a_1+b_1+c_1}{3}$$

$$\iff \left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2 \geq 3\cdot\left(\frac{a+\frac{1}{a}+b+\frac{1}{b}+c+\frac{1}{c}}{3}\right)^2$$

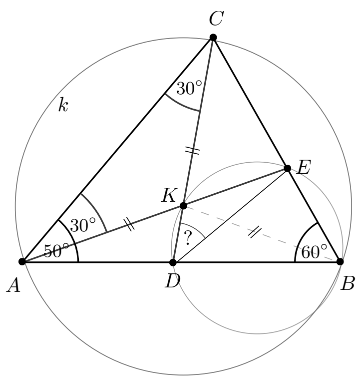
$$\iff \left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2 \geq \frac{1}{3}\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2. \qquad (*)$$

Using the inequality between the harmonic and the arithmetic mean for a, b, c we get

$$\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \leq \frac{a+b+c}{3} = \frac{1}{3} \implies \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9. \qquad (**)$$

Now we plug-in $(**)$ to $(*)$ and get $\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2 \geq \frac{1}{3}(1+9)^2 = \frac{100}{3}$.
Equality holds if and only if $a=b=c=1/3$. ▲

10. Let ABC be a triangle with $\sphericalangle BAC = 50^\circ$, $\sphericalangle ABC = 60^\circ$. If D and E are points on sides AB and BC, respectively, so that $\sphericalangle DCA = \sphericalangle EAC = 30^\circ$, compute the measure of angle $\sphericalangle CDE$.



Solution. Let $AE \cap CD = \{K\}$. Triangle AKC is isosceles since $\sphericalangle KAC = \sphericalangle KCA = 30^\circ$. It follows that $AK = KC$ and $\sphericalangle AKC = 120^\circ$. Since $\sphericalangle AKC = 2\cdot\sphericalangle ABC$ and $KA = KC$, we conclude that K is the centre of circumcircle k of $\triangle ABC$.

Since $AK = KC = KB$, we obtain that $20^\circ = \sphericalangle KAB = \sphericalangle KBD$ and, since vertical angles are congruent, $\sphericalangle DKE = \sphericalangle AKC = 120^\circ$. From $\sphericalangle DKE + \sphericalangle DBE = 180^\circ$ it follows that rectangle $DBEK$ is cyclic. Since all angles inscribed in a circle and subtended by the same chord are equal, we obtain that $\sphericalangle KDE = \sphericalangle KBE = 40^\circ$. Therefore,

$$\sphericalangle CDE = \sphericalangle KDE = 40^\circ. \qquad ▲$$

11. Find all positive integers m and n so that $2^n = 3^m + 5$.

Solution. For $n=3$ we get that $m=1$, and for $n=5$, $m=3$. We will prove that these are the only possible solutions.

One can easily check that $n \in \{1, 2, 4\}$ cannot be a solution. Suppose now that $n > 5$. In this case 2^n is divisible by 64 and all possible remainders r when dividing 3^k by 64 are given below:

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
r	3	9	27	17	51	25	11	33	35	41	59	49	19	57	43	1

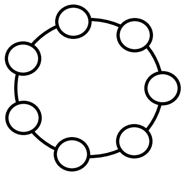
Since $64 \mid 3^m + 5$, we conclude that $m = 11 + 16\ell$, for $\ell \in \mathbb{N}$. Now let us show that, in this case, the remainder when dividing 2^n by 17 is not the same as when dividing $3^m + 5$ by 17.

We have that $3^m = (3^{16})^{\ell} \cdot 3^{11}$ and, since $3^{16} \equiv_{17} 1$ (Fermat's little theorem) and $3^{11} \equiv_{17} 7$, we conclude that $3^m + 5 \equiv_{17} 12$. On the other hand, the possible remainders r when dividing 2^n by 17 are given below:

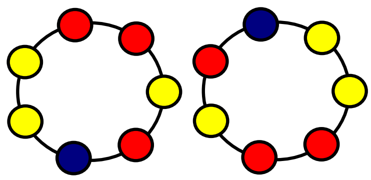
n	1	2	3	4	5	6	7	8
r	2	4	8	16	15	13	9	1

Number 12 does not appear in the table, which proves that $n > 5$ cannot be a solution. Therefore, the only solutions are $n = 3, m = 1$, and $n = 5, m = 3$. ▲

12. We are given a necklace with 7 beads, as in the figure below. Each of the beads we colour in one of the three colours (red, blue, yellow). We say that such a colouring is *colourful* if every colour is used at least once. How many different colourful colourings are there, if the colourings that could be obtained one from another by rotating the necklace are considered to be the same?

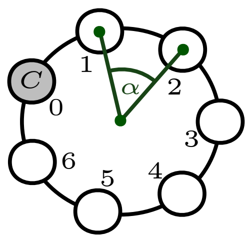


Necklace before colouring.



Two colourful colourings that are the same.

Solution. First we enumerate the beads with numbers $0 - 6$, as shown in the picture. Then each of the seven beads can be coloured in one of the 3 colours, which gives us 3^7 configurations. Two problems arise: (1) How many of those configurations are not colourful? (2) How many of those did we count more than once, because they are the same with respect to a rotation (allowed rotations are $k \cdot \frac{360^\circ}{7} = k\alpha$, for $k \in \{1, 2, 3, 4, 5, 6\}$)?



Let us first answer the question number (2). The key question to answer is what configurations are invariant under rotation for the angle $k\alpha$. In other words, how does a necklace have to be coloured so that, when we rotate it for $k\alpha$, where k is a fixed number from $\{1, 2, 3, 4, 5, 6\}$, we get the same necklace? Let us colour the bead numbered with 0 in colour C (C is one of Yellow, Blue, Red), and suppose we get the same necklace when rotating for $k\alpha$, for some $k \in \{1, 2, 3, 4, 5, 6\}$. Then the bead k also has to be coloured in C, as well as the beads numbered with $(2k)_{\text{mod } 7}$, $(3k)_{\text{mod } 7}$, $(4k)_{\text{mod } 7}$, $(5k)_{\text{mod } 7}$, and $(6k)_{\text{mod } 7}$. Since 7 is a prime number, we get that

$$\{0, k, (2k)_{\text{mod } 7}, (3k)_{\text{mod } 7}, (4k)_{\text{mod } 7}, (5k)_{\text{mod } 7}, (6k)_{\text{mod } 7}\} = \{0, 1, 2, 3, 4, 5, 6\},$$

i.e., all beads have to be coloured in colour C. There are 3 monochrome necklaces, therefore the number of *different* necklaces with *at most* 3 colours is equal to

$$YBR = \underbrace{\frac{1}{7}(3^7 - 3)}_{\substack{7\,\textit{different}\text{ config's} \\ \text{give the same necklace}}} + \underbrace{3}_{\substack{\text{monochromatic} \\ \text{necklaces}}} = \frac{3}{7} \cdot (3^3 - 1) \cdot (3^3 + 1) + 3 = 315.$$

We can use the same reasoning to conclude that the number of *different* necklaces with *at most* 2 colours is equal to $\frac{1}{7}(2^7 - 2) + 2 = 20$, and for one colour this number is 1. Using the Inclusion-Exclusion Principle, we get that the number of *different colourful* necklaces (necklaces with *exactly* 3 colours) is equal to:

$$YBR - YR - YG - GR + Y + G + R = 315 - 3 \cdot 20 + 3 = 258.$$ ▲