SOLUTIONS TO THE

MATHEMATICAL GRAMMAR SCHOOL CUP

- MATHEMATICS -

1. B); **2.** A); **3.** D); **4.** E); **5.** C); **6.** C); **7.** D); **8.** C).

9. For any integer a one of the following statements is true:

$a \equiv 0 \pmod 9$;
$a \equiv \pm 1 \pmod 9$;
$a \equiv \pm 2 \pmod 9$;
$a \equiv \pm 3 \pmod 9$;
$a \equiv \pm 4 \pmod 9$;

so we conclude that for any integer a on of the following holds:

$a^3 \equiv 0 \pmod 9$;
$a^3 \equiv \pm 1 \pmod 9$;
$a^3 \equiv \mp 1 \pmod 9$;
$a^3 \equiv 0 \pmod 9$;
$a^3 \equiv \pm 1 \pmod 9$;

so all possible remainders in the division of integer a^3 by 9 are $0, 1, -1$. On the other side, it holds that $2000 \equiv 2 \pmod 9$ so $2000^2 \equiv 4 \pmod 9$, and the sum of three cubes cannot give 4 as the remainder in division by 9. Hence, there are no integers x, y, z satisfying the given equality.

10. Let E and D be the points where the lines parallel to the sides BC and AB of the triangle ABC intersect the side AC of the triangle ABC. Smaller triangles are similar to the triangle ABC so it holds:

$$\sqrt{\frac{P_1}{P}} = \frac{ED}{AC}, \ \sqrt{\frac{P_2}{P}} = \frac{AD}{AC}, \ \sqrt{\frac{P_1}{P}} = \frac{EC}{AC}.$$

If we add these equalities we get

$$\sqrt{\frac{P_1}{P}} + \sqrt{\frac{P_2}{P}} + \sqrt{\frac{P_3}{P}} = \frac{ED + AD + EC}{AC} = 1,$$

which implies that $\sqrt{P} = \sqrt{P_1} + \sqrt{P_2} + \sqrt{P_3}$, so $P = (\sqrt{P_1} + \sqrt{P_2} + \sqrt{P_3})^2$.

11. According to the inequality between the arithmetic mean and the geometric mean it holds:

$$ab + \frac{b}{a} \geq 2\sqrt{ab \cdot \frac{b}{a}} = 2b; \tag{1}$$

$$ab + \frac{a}{b} \geq 2\sqrt{ab \cdot \frac{a}{b}} = 2a; \tag{2}$$

$$\frac{b}{a} + \frac{a}{b} \geq 2\sqrt{\frac{b}{a} \cdot \frac{a}{b}} = 2. \tag{3}$$

Now adding inequalities (1), (2) and (3) gives us:

$$2(ab + \tfrac{a}{b} + \tfrac{b}{a}) \geq 2(a + b + 1).$$

The equality holds if and only if $a = b = 1$.

12. $$\begin{aligned} p^{p+2} + (p+2)^p &\equiv p^{p+2} + (-p)^p (\text{mod } 2p+2) \\ &\equiv p^{p+2} - p^p (\text{mod } 2p+2) \\ &\equiv p^p(p^2-1)(\text{mod } 2p+2) \\ &\equiv p^p(p-1)(p+1)(\text{mod } 2p+2) \\ &\equiv 0 (\text{mod } 2p+2) \end{aligned}$$
because $2p+2|(p-1)(p+1)$ $(2|p-1$ and $p+1|p+1)$.