SOLUTIONS TO THE MATHEMATICAL GRAMMAR SCHOOL CUP - MATHEMATICS -

1\. B) 2. C) 3. C) 4. A) 5. B) 6. D) 7. B) 8. E)

9\. Let us denote

$$x = m - 3,\ y = n - 5,\ z = p - 7.$$

Now we have $x + y + z = 0$ and $x^3 + y^3 + z^3 = 540$. Since

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx),$$

we have that $xyz = 180 = 2^2 \cdot 3^2 \cdot 5$, so the solutions are

$(m, n, p) \in \{(-1, 0, 16), (-2, 1, 16), (12, 0, 3), (12, 1, 2)(-2, 14, 3), (-1, 14, 2)\}$

10\. From the AM-GM inequality (inequality of arithmetic and geometric means) we have $x^2 + 2yz \leq x^2 + y^2 + z^2$, hence

$$\frac{x^2}{x^2 + 2yz} \geq \frac{x^2}{x^2 + y^2 + z^2} \tag{1}$$

In the same way we would get

$$\frac{y^2}{y^2 + 2xz} \geq \frac{y^2}{x^2 + y^2 + z^2} \tag{2}$$

and

$$\frac{z^2}{z^2 + 2xy} \geq \frac{z^2}{x^2 + y^2 + z^2}. \tag{3}$$

Adding inequalities (1), (2), (3) gives us

$$\frac{x^2}{x^2 + 2yz} + \frac{y^2}{y^2 + 2xz} + \frac{z^2}{z^2 + 2xy} \geq \frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2} = 1 \tag{4}$$

Noticing that

$$\frac{x^2}{x^2 + 2yz} + \frac{y^2}{y^2 + 2zx} + \frac{z^2}{z^2 + 2xy} + 2 \cdot \left(\frac{yz}{x^2 + 2yz} + \frac{zx}{y^2 + 2zx} + \frac{xy}{z^2 + 2xy} \right) = 3$$

together with (4) implies

$$2 \cdot \left(\frac{yz}{x^2 + 2yz} + \frac{zx}{y^2 + 2zx} + \frac{xy}{z^2 + 2xy} \right) \leq 2.$$

Hence,

$$\frac{yz}{x^2 + 2yz} + \frac{zx}{y^2 + 2zx} + \frac{xy}{z^2 + 2xy} \leq 1.$$

The equality holds if and only if $x = y = z$.

11. The base of the pyramid is a triangle, so the length of its altitude is $h = \sqrt{3^2 - 2^2} = \sqrt{5}$, while its area is $B = \dfrac{4\sqrt{5}}{2} = 2\sqrt{5}$.
Further, the radius of the circle circumscribed about the base is

$$R = \frac{abc}{4B} = \frac{3 \cdot 3 \cdot 4}{4 \cdot 2 \cdot \sqrt{5}} = \frac{9}{2 \cdot \sqrt{5}}$$

The foot of the altitude of the pyramid is the center of the circle circumscribed about the base, hence

$$H = \sqrt{3^2 - R^2} = \sqrt{9 - \Big(\frac{9}{2 \cdot \sqrt{5}}\Big)^2} = \frac{3}{2}\sqrt{\frac{11}{5}}.$$

The volume of the pyramid is $V = \dfrac{1}{3} B \cdot H = \dfrac{1}{3} 2\sqrt{5} \cdot \dfrac{3}{2}\sqrt{\dfrac{11}{5}} = \sqrt{11}$.

12. Since $10^4 \equiv 4 \pmod 7$ and $10^6 \equiv 1 \pmod 7$, we have

$$10^{6k+4} + 3 \equiv 0 \pmod 7.$$

Hence, $7 \mid 10^{6k+4} + 3$.

Remark. Similarly we would show that $13 \mid 10^{6k+1} + 3$.